Orthogonal Cayley-Sudoku Tables

Michael Ward
Western Oregon University

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Speaker's Qualifications

Cayley-Sudoku Tables–Expert

Latin Squares & Orthogonality–Dilettante

Latin Squares and Orthogonal Mates

Definition

*A **Latin square** is an $n \times n$ array in which each of n symbols appears exactly once in each row and in each column.*

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*Two Latin squares are **orthogonal**, or are **orthogonal mates**, provided each ordered pair of symbols occurs exactly once when the squares are superimposed.*

The notion dates back to the time of Euler, at least.

Non-example

0	1	2
1	2	0
2	0	1

0	2	1
2	1	0
1	0	2

$\boxed{00}$	12	21
12	21	$\boxed{00}$
21	$\boxed{00}$	12

Non-example

0	1	2
1	2	0
2	0	1

0	2	1
2	1	0
1	0	2

$\boxed{00}$	12	21
12	21	$\boxed{00}$
21	$\boxed{00}$	12

Example

0	1	2
1	2	0
2	0	1

0	1	2
2	0	1
1	2	0

00	11	22
12	20	01
21	02	10

An Old Question

Theorem

The Cayley (i.e. operation) table of any finite group is a (bordered) Latin square.

Which Cayley tables have orthogonal mates?

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Which Cayley tables have orthogonal mates?

Example

The Cayley tables of $\mathbb{Z}_3 := \{0, 1, 2\}$ *under addition modulo 3 from the previous slide*

	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

	0	1	2
0	0	1	2
2	2	0	1
1	1	2	0

Answers to the Old Question

Definition

A ***complete mapping*** *of a group* G *is a bijection* $\theta : G \to G$ *for which the mapping* $\eta : G \to G$ *where* $\eta(x) = x\theta(x)$ *is also a bijection.*

Answers to the Old Question

Definition

A ***complete mapping*** *of a group G is a bijection $\theta : G \to G$ for which the mapping $\eta : G \to G$ where $\eta(x) = x\theta(x)$ is also a bijection.*

Remember this–

Theorem

The Cayley table of a finite group G has an orthogonal mate iff G has a complete mapping.

Answers to the Old Question

Definition

*A **complete mapping** of a group G is a bijection $\theta : G \to G$ for which the mapping $\eta : G \to G$ where $\eta(x) = x\theta(x)$ is also a bijection.*

Remember this–

Theorem

The Cayley table of a finite group G has an orthogonal mate iff G has a complete mapping.

A digression–

Hall-Paige ~~Conjecture~~ Theorem (circa 2009) *A finite group G has a complete map iff G has trivial or non-cyclic Sylow 2-subgroups.*

Cayley-Sudoku Tables

Definition

*A **Sudoku table** is an $n \times n$ array partitioned into rectangular blocks of some fixed size such that each of n symbols appear exactly once in each row, each column, and each block.*

(In the ubiquitous Sudoku puzzle, the array is 9×9; the blocks are 3×3; and the symbols are the numbers 1 through 9.)

Definition

*A **Cayley-Sudoku table** is a Cayley table which is also a (bordered) Sudoku table.*

	0	3	6	1	4	7	2	5	8
0	0	3	6	1	4	7	2	5	8
1	1	4	7	2	5	8	3	6	0
2	2	5	8	3	6	0	4	7	1
3	3	6	0	4	7	1	5	8	2
4	4	7	1	5	8	2	6	0	3
5	5	8	2	6	0	3	7	1	4
6	6	0	3	7	1	4	8	2	5
7	7	1	4	8	2	5	0	3	6
8	8	2	5	0	3	6	1	4	7

A Cayley-Sudoku Table for $\mathbb{Z}_9 := \{0,1,2,3,4,5,6,7,8\}$ under addition mod 9

	()	(1,2,3,4)	(1,3)(2,4)	(1,4,3,2)	(1,4)(2,3)	(1,3)	(1,2)(3,4)	(2,4)
() (1,2)(3,4)	() (1,2)(3,4)	(1,2,3,4) (1,3)	(1,3)(2,4) (1,4)(2,3)	(1,4,3,2) (2,4)	(1,4)(2,3) (1,3)(2,4)	(1,3) (1,2,3,4)	(1,2)(3,4) ()	(2,4) (1,4,3,2)
(1,2,3,4) (2,4)	(1,2,3,4) (2,4)	(1,3)(2,4) (1,2)(3,4)	(1,4,3,2) (1,3)	() (1,4)(2,3)	(1,3) (1,4,3,2)	(1,2)(3,4) (1,3)(2,4)	(2,4) (1,2,3,4)	(1,4)(2,3) ()
(1,3)(2,4) (1,4)(2,3)	(1,3)(2,4) (1,4)(2,3)	(1,4,3,2) (2,4)	() (1,2)(3,4)	(1,2,3,4) (1,3)	(1,2)(3,4) ()	(2,4) (1,4,3,2)	(1,4)(2,3) (1,3)(2,4)	(1,3) (1,2,3,4)
(1,4,3,2) (1,3)	(1,4,3,2) (1,3)	() (1,4)(2,3)	(1,2,3,4) (2,4)	(1,3)(2,4) (1,2)(3,4)	(2,4) (1,2,3,4)	(1,4)(2,3) ()	(1,3) (1,4,3,2)	(1,2)(3,4) (1,3)(2,4)

A Cayley-Sudoku Table for D_4, dihedral group of order 8

A New Question?

Which Cayley-Sudoku tables have orthogonal mates that are also Cayley-Sudoku tables (with the same size blocks)?

Examples–from a different perspective–were given for the additive groups of certain finite fields by Pedersen & Vis (*College Math. J.* 2009) and by Lorch (*Amer. Math. Monthly* 2012).

A New Example

	0	3	6	1	4	7	2	5	8
0	0	3	6	1	4	7	2	5	8
1	1	4	7	2	5	8	3	6	0
2	2	5	8	3	6	0	4	7	1
3	3	6	0	4	7	1	5	8	2
4	4	7	1	5	8	2	6	0	3
5	5	8	2	6	0	3	7	1	4
6	6	0	3	7	1	4	8	2	5
7	7	1	4	8	2	5	0	3	6
8	8	2	5	0	3	6	1	4	7

	0	3	6	1	4	7	2	5	8
0	0	3	6	1	4	7	2	5	8
8	8	2	5	0	3	6	1	4	7
7	7	1	4	8	2	5	0	3	6
6	6	0	3	7	1	4	8	2	5
5	5	8	2	6	0	3	7	1	4
4	4	7	1	5	8	2	6	0	3
3	3	6	0	4	7	1	5	8	2
2	2	5	8	3	6	0	4	7	1
1	1	4	7	2	5	8	3	6	0

Orthogonal Cayley-Sudoku Tables for $\mathbb{Z}_9$

Another New Example

	()	(1,2,3,4)	(1,3)(2,4)	(1,4,3,2)	(1,4)(2,3)	(1,3)	(1,2)(3,4)	(2,4)
() (1,2)(3,4)	() (1,2)(3,4)	(1,2,3,4) (1,3)	(1,3)(2,4) (1,4)(2,3)	(1,4,3,2) (2,4)	(1,4)(2,3) (1,3)(2,4)	(1,3) (1,2,3,4)	(1,2)(3,4) ()	(2,4) (1,4,3,2)
(1,2,3,4) (2,4)	(1,2,3,4) (2,4)	(1,3)(2,4) (1,2)(3,4)	(1,4,3,2) (1,3)	() (1,4)(2,3)	(1,3) (1,4,3,2)	(1,2)(3,4) (1,3)(2,4)	(2,4) (1,2,3,4)	(1,4)(2,3) ()
(1,3)(2,4) (1,4)(2,3)	(1,3)(2,4) (1,4)(2,3)	(1,4,3,2) (2,4)	() (1,2)(3,4)	(1,2,3,4) (1,3)	(1,2)(3,4) ()	(2,4) (1,4,3,2)	(1,4)(2,3) (1,3)(2,4)	(1,3) (1,2,3,4)
(1,4,3,2) (1,3)	(1,4,3,2) (1,3)	() (1,4)(2,3)	(1,2,3,4) (2,4)	(1,3)(2,4) (1,2)(3,4)	(2,4) (1,2,3,4)	(1,4)(2,3) ()	(1,3) (1,4,3,2)	(1,2)(3,4) (1,3)(2,4)

	()	(1,2,3,4)	(1,3)(2,4)	(1,4,3,2)	(1,4)(2,3)	(1,3)	(1,2)(3,4)	(2,4)
() (2,4)	() (2,4)	(1,2,3,4) (1,2)(3,4)	(1,3)(2,4) (1,3)	(1,4,3,2) (1,4)(2,3)	(1,4)(2,3) (1,4,3,2)	(1,3) (1,3)(2,4)	(1,2)(3,4) (1,2,3,4)	(2,4) ()
(1,4,3,2) (1,2)(3,4)	(1,4,3,2) (1,2)(3,4)	() (1,3)	(1,2,3,4) (1,4)(2,3)	(1,3)(2,4) (2,4)	(2,4) (1,3)(2,4)	(1,4)(2,3) (1,2,3,4)	(1,3) ()	(1,2)(3,4) (1,4,3,2)
(1,4)(2,3) (1,2,3,4)	(1,4)(2,3) (1,2,3,4)	(2,4) (1,3)(2,4)	(1,2)(3,4) (1,4,3,2)	(1,3) ()	() (1,3)	(1,4,3,2) (1,2)(3,4)	(1,3)(2,4) (2,4)	(1,2,3,4) (1,4)(2,3)
(1,3) (1,3)(2,4)	(1,3) (1,3)(2,4)	(1,4)(2,3) (1,4,3,2)	(2,4) ()	(1,2)(3,4) (1,2,3,4)	(1,2,3,4) (1,2)(3,4)	() (2,4)	(1,4,3,2) (1,4)(2,3)	(1,3)(2,4) (1,3)

Orthogonal Cayley-Sudoku Tables for D_4, dihedral group of order 8

How the original $\mathbb{Z}_9$ Cayley-Sudoku table was made: Column Labels

	0	3	6	1	4	7	2	5	8
0	0	3	6	1	4	7	2	5	8
1	1	4	7	2	5	8	3	6	0
2	2	5	8	3	6	0	4	7	1
3	3	6	0	4	7	1	5	8	2
4	4	7	1	5	8	2	6	0	3
5	5	8	2	6	0	3	7	1	4
6	6	0	3	7	1	4	8	2	5
7	7	1	4	8	2	5	0	3	6
8	8	2	5	0	3	6	1	4	7

Consider the subgroup $\langle 3 \rangle = \{0,3,6\}$. The column labels of each block $[0,3,6] = [0,3,6] + 0$, $[1,4,7] = [0,3,6] + 1$ and $[2,5,8] = [0,3,6] + 2$ are right cosets of the subgroup.

Row Labels

	0	3	6	1	4	7	2	5	8
0	0	3	6	1	4	7	2	5	8
1	1	4	7	2	5	8	3	6	0
2	2	5	8	3	6	0	4	7	1
3	3	6	0	4	7	1	5	8	2
4	4	7	1	5	8	2	6	0	3
5	5	8	2	6	0	3	7	1	4
6	6	0	3	7	1	4	8	2	5
7	7	1	4	8	2	5	0	3	6
8	8	2	5	0	3	6	1	4	7

The row labels of each block are complete sets of left coset representatives, i.e. one element from each left coset.

$0 + [0,3,6] = [0,3,6]$

$1 + [0,3,6] = [1,4,7]$

$2 + [0,3,6] = [2,5,8]$

Generalization

Theorem (J. Dénes 1967; J. Carmichael, K. Schloeman, M. Ward 2010)

Let G be a finite group. Assume H is a subgroup of G. Let $Hg_1, Hg_2, \ldots, Hg_n$ be the distinct right cosets of H in G and let $T_1, T_2, \ldots, T_k$ partition G into complete sets of left coset representatives [CSLCR] of H in G. The following layout gives a Cayley-Sudoku table.

	Hg_1	Hg_2	$\ldots$	Hg_n
T_1				
T_2				
$\vdots$				
T_k				

Cayley-Sudoku tables exist for every group using any subgroup.

How the orthogonal $\mathbb{Z}_9$ Cayley-Sudoku table was made

Leave the column labels unchanged.
The map $\theta(x) = x$ is a complete map for $\mathbb{Z}_9$.
Apply θ to each row label and then take the inverse.

	0	3	6	1	4	7	2	5	8
0	0	3	6	1	4	7	2	5	8
1	1	4	7	2	5	8	3	6	0
2	2	5	8	3	6	0	4	7	1
3	3	6	0	4	7	1	5	8	2
4	4	7	1	5	8	2	6	0	3
5	5	8	2	6	0	3	7	1	4
6	6	0	3	7	1	4	8	2	5
7	7	1	4	8	2	5	0	3	6
8	8	2	5	0	3	6	1	4	7

	0	3	6	1	4	7	2	5	8
$-\theta(0) = -0 = 0$	0	3	6	1	4	7	2	5	8
$-\theta(1) = -1 = 8$	8	2	5	0	3	6	1	4	7
$-\theta(2) = -2 = 7$	7	1	4	8	2	5	0	3	6
$-\theta(3) = -3 = 6$	6	0	3	7	1	4	8	2	5
$-\theta(4) = -4 = 5$	5	8	2	6	0	3	7	1	4
$-\theta(5) = -5 = 4$	4	7	1	5	8	2	6	0	3
$-\theta(6) = -6 = 3$	3	6	0	4	7	1	5	8	2
$-\theta(7) = -7 = 2$	2	5	8	3	6	0	4	7	1
$-\theta(8) = -8 = 1$	1	4	7	2	5	8	3	6	0

Same construction worked for Cayley-Sudoku table of D_4

Leave the column labels unchanged.

Hall & Paige gave a complete map θ for D_4. (Not the identity map!)

Apply θ to each row label and then take the inverse.

	()	(1,2,3,4)	(1,3)(2,4)	(1,4,3,2)	(1,4)(2,3)	(1,3)	(1,2)(3,4)	(2,4)
() (1,2)(3,4)	() (1,2)(3,4)	(1,2,3,4) (1,3)	(1,3)(2,4) (1,4)(2,3)	(1,4,3,2) (2,4)	(1,4)(2,3) (1,3)(2,4)	(1,3) (1,2,3,4)	(1,2)(3,4) ()	(2,4) (1,4,3,2)
(1,2,3,4) (2,4)	(1,2,3,4) (2,4)	(1,3)(2,4) (1,2)(3,4)	(1,4,3,2) (1,3)	() (1,4)(2,3)	(1,3) (1,4,3,2)	(1,2)(3,4) (1,3)(2,4)	(2,4) (1,2,3,4)	(1,4)(2,3) ()
(1,3)(2,4) (1,4)(2,3)	(1,3)(2,4) (1,4)(2,3)	(1,4,3,2) (2,4)	() (1,2)(3,4)	(1,2,3,4) (1,3)	(1,2)(3,4) ()	(2,4) (1,4,3,2)	(1,4)(2,3) (1,3)(2,4)	(1,3) (1,2,3,4)
(1,4,3,2) (1,3)	(1,4,3,2) (1,3)	() (1,4)(2,3)	(1,2,3,4) (2,4)	(1,3)(2,4) (1,2)(3,4)	(2,4) (1,2,3,4)	(1,4)(2,3) ()	(1,3) (1,4,3,2)	(1,2)(3,4) (1,3)(2,4)

	()	(1,2,3,4)	(1,3)(2,4)	(1,4,3,2)	(1,4)(2,3)	(1,3)
$\theta(())^{-1} = ()^{-1} = ()$ $\theta((1,2)(3,4))^{-1} = (2,4)^{-1} = (2,4)$	() (2,4)	(1,2,3,4) (1,2)(3,4)	(1,3)(2,4) (1,3)	(1,4,3,2) (1,4)(2,3)	(1,4)(2,3) (1,4,3,2)	(1,3) (1,3)(2,4)
$\theta((1,2,3,4))^{-1} = (1,2,3,4)^{-1} = (1,4,3,2)$ $\theta((2,4))^{-1} = ((1,2)(3,4))^{-1} = (1,2)(3,4)$	(1,4,3,2) (1,2)(3,4)	() (1,3)	(1,2,3,4) (1,4)(2,3)	(1,3)(2,4) (2,4)	(2,4) (1,3)(2,4)	(1,4)(2,3) (1,2,3,4)
$\theta((1,3)(2,4))^{-1} = ((1,4)(2,3))^{-1} = (1,4)(2,3)$ $\theta((1,4)(2,3))^{-1} = (1,4,3,2)^{-1} = (1,2,3,4)$	(1,4)(2,3) (1,2,3,4)	(2,4) (1,3)(2,4)	(1,2)(3,4) (1,4,3,2)	(1,3) ()	() (1,3)	(1,4,3,2) (1,2)(3,4)
$\theta((1,4,3,2))^{-1} = (1,3)^{-1} = (1,3)$ $\theta((1,3))^{-1} = ((1,3)(2,4))^{-1} = (1,3)(2,4)$	(1,3) (1,3)(2,4)	(1,4)(2,3) (1,4,3,2)	(2,4) ()	(1,2)(3,4) (1,2,3,4)	(1,2,3,4) (1,2)(3,4)	() (2,4)

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  (The row labels in the new table must turn out to be CSLCR.)

Generalization

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- Applying θ-inverse to any Cayley-Sudoku table always gives orthogonal *Cayley* tables.
  (Follows readily from the definition of a complete map. Nothing new about that.)
- But . . . not always orthogonal Cayley-*Sudoku* tables
  (The row labels in the new table must turn out to be CSLCR.)
- However . . . So far, for any group having a complete mapping there exists a (carefully chosen) Cayley-Sudoku table for which θ-inverse yields orthogonal Cayley-Sudoku tables.

Questions

- For any group having a complete mapping does there exists a Cayley-Sudoku table for which the θ-inverse construction yields an orthogonal mate?

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- How many mutually orthogonal Cayley-Sudoku tables can a group have?

Questions

- For any group having a complete mapping does there exists a Cayley-Sudoku table for which the θ-inverse construction yields an orthogonal mate?
- Is there a characterization of groups and subgroups for which orthogonal Cayley-Sudoku tables (based on that subgroup) exist?
- How many mutually orthogonal Cayley-Sudoku tables can a group have?
- Is there a Latin square expert who can tell me what I'm talking about?

Bonus Slide

	00	12	21	11	20	02	22	01	10
00	00	12	21	11	20	02	22	01	10
20	20	02	11	01	10	22	12	21	00
10	10	22	01	21	00	12	02	11	20
12	12	21	00	20	02	11	01	10	22
02	02	11	20	10	22	01	21	00	12
22	22	01	10	00	12	21	11	20	02
21	21	00	12	02	11	20	10	22	01
11	11	20	02	22	01	10	00	12	21
01	01	10	22	12	21	00	20	02	11

	00	12	21	11	20	02	22	01	10
00	00	12	21	11	20	02	22	01	10
10	10	22	01	21	00	12	02	11	20
20	20	02	11	01	10	22	12	21	00
21	21	00	12	02	11	20	10	22	01
01	01	10	22	12	21	00	20	02	11
11	11	20	02	22	01	10	00	12	21
12	12	21	00	20	02	11	01	10	22
22	22	01	10	00	12	21	11	20	02
02	02	11	20	10	22	01	21	00	12

Orthogonal *Magic* Cayley-Sudoku tables for $\mathbb{Z}_3 \times \mathbb{Z}_3$

(Hear Rosanna Mersereau's talk at this Meeting.)