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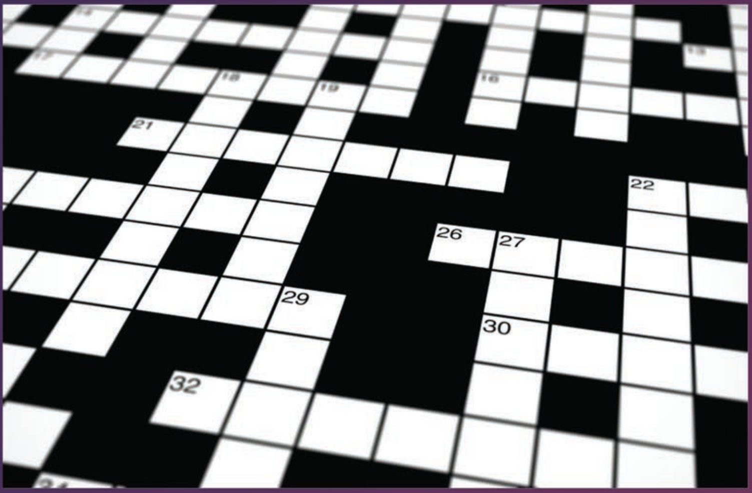
Crossword Counting Conundrums



Ben Coté, Angel Cruz, Leanne Merrill, Brandilan Moring, TJ Rabosky, and Haley Willmott

Crossword puzzles are all around us: in the middle of the local newspaper, on the shelves of every dollar store, and on countless apps and websites. Crosswords come in a variety of shapes, sizes, and difficulty levels. Whether or not you're a cruciverbalist—a lover of crosswords—you can find a lot of mathematics within the interlocking wordplay of these puzzles.

What Is a Crossword Puzzle?

For purposes of this article, a crossword puzzle is a square grid of squares, each of which is either black (a *void*) or white (a *cell*). A maximal vertical sequence of adjacent cells creates a *Down answer,* whereas an *Across answer* consists of a maximal horizontal sequence of cells.

Crossword puzzle style differs by country and type of media. Herein, we restrict our study to the most common type of crossword puzzle in the United States: American-style crosswords. If you open a copy of the *New York Times*, this is the type of crossword puzzle you will most likely find.

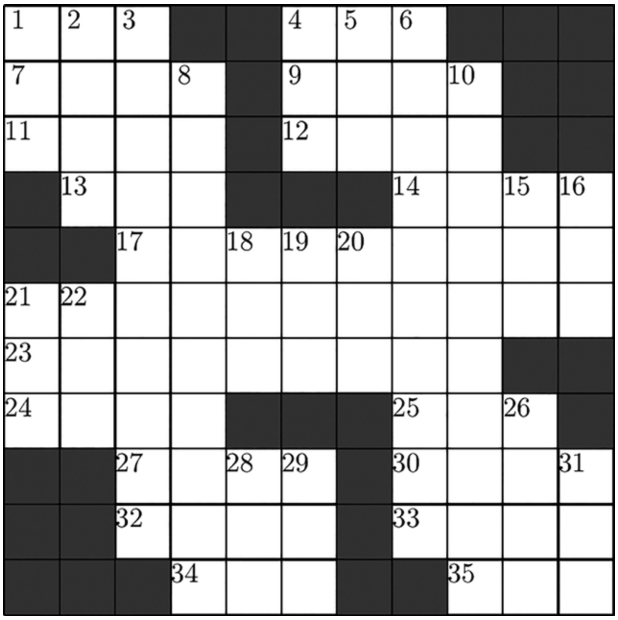
An *American-style crossword puzzle* satisfies the following structure rules:

1. Any cell can be reached from every cell by traversing the puzzle using only horizontal and vertical steps and only passing through cells. In other words, the voids do not "break up" the puzzle into smaller, disconnected portions.
2. The puzzle possesses 180-degree rotational symmetry: The puzzle will appear unchanged if it is rotated 180 degrees.
3. Each answer must be at least three cells long.
4. Each cell must be part of both a Down and an Across answer.
5. There is not a full row or full column of voids.

Figure 1 shows a valid 11 × 11 American-style crossword grid.

Along with the grid, American-style crossword puzzles include a list of *Across* and *Down clues* that are used to fill in the cells of the grid, usually with a single letter each. *Solving* the puzzle requires one to correctly fill in the letters based on the clues.

Figure 1. An American-style crossword grid.



There are numerous interesting questions to be asked about the "word" part of crosswords, and, as such, crossword puzzles are of interest in computational linguistics and natural language processing research. In fact, crossword puzzle solving has emerged as a benchmark for machine learning that involves language and word meaning (Kulshreshtha et al. "Down and Across: Introducing Crossword-Solving as a New NLP Benchmark." *Proc. of the 60th Annual Meeting of the ACL*. Vol 1. [2022] 2648–2659). But here we are primarily interested in the mathematics behind the grids themselves, so we will ignore the clues.

Some Crossword Grid Questions

The listed structure rules of an American-style crossword puzzle create restrictions on the number and types of allowed grids. As a result, many mathematical questions arise:

1. How many $n \times n$ crossword puzzle grids exist?
2. How many $n \times n$ crossword puzzle grids possess 90-degree symmetry?
3. What is the maximum number of answers that can fit in a 15×15 grid?
4. What is the maximum number of voids in a 15×15 crossword puzzle?
5. How many different puzzles achieve the maximum number of voids?

These questions only scratch the surface of mathematically inspired questions about crossword puzzle grids, and surprisingly few such questions have been answered. Kevin Ferland determined that 96 is the maximum number of answers that fit in a 15×15 grid ("Record Crossword Puzzles." *Amer. Math. Monthly*. 121(6) [2017] 534–536). Interestingly, the solution is not unique; there are multiple grids that contain 96 answers. Four authors of this paper (Cruz, Moring, Rabosky, and Willmott) investigated some of these remaining questions as part of their senior capstone projects at Western Oregon University.

Question 1 can be readily investigated for small n by hand. For $n = 3$, 4, and 5. The answers are 1, 3, and 12, respectively. As n grows, this problem gets more challenging. For example, when $n = 6$, there are 46 puzzle grids, and for $n = 7$, the number increases to 312 (Cruz). The *FiveThirtyEight* "Riddler" column posed this question for the most common puzzle size, $n = 15$, in 2019 (Roeder. "How Many Crossword Puzzles Can You Make?"). In response, Jim Ferry used dynamic programming to generate 404,139,015,237,875 different such grids (*github.com/jimferry/crosswords-538*). There is, however, no known general formula for the number of puzzles in an arbitrary grid size.

Grids with 90-degree symmetry have been enumerated for odd dimensions up through $n = 9$: There are two different 5×5 puzzles, 18 distinct 7×7 puzzles, and 183 unique 9×9 puzzles (Moring).

To take an initial crack at Question 4, one can quickly verify that in a 3×3 puzzle, the maximum number of voids is 0. We will show that for any larger dimension, n, the maximum number of voids is given by $n^2 - 3n - 2$ (Rabosky, Willmott). In other words, the minimum number of cells in a valid puzzle is $3n + 2$.

Finding a Lower Bound on Cells

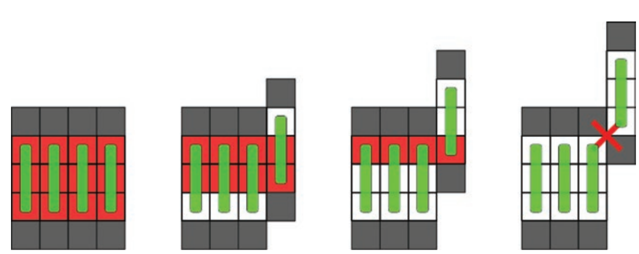
The structure rules require that all rows and columns contain at least three cells. Therefore, an $n \times n$ puzzle grid must have at least $3n$ cells. All that's left to justify is the necessary presence of the extra two cells.

Let's first assume that there are only $3n$ cells, which forces there to be exactly three adjacent cells in each row and column of the puzzle by the answer length rule. Consider the leftmost column of the puzzle, which has three adjacent cells. Regardless of the vertical position of those cells, the next two columns to the right also have three cells at precisely the same vertical position as the left-column cells. These cells thus form a 3×3 block along the left side of the puzzle; any other configuration in these three columns creates a word of length 1 or 2.

If the puzzle is 3×3, then this 3×3 block of cells gives the unique grid satisfying the structure rules for this size. If the puzzle is larger, then none of the three cells in the fourth column can be at the same vertical position as those in the first three columns, as that would produce a row with four cells, contradicting the assumption that there are precisely $3n$ cells in the puzzle (see figure 2). Therefore, the fourth column cells must be offset from those in the first three columns. However, any such configuration violates the connectivity condition required of the puzzle (see figure 2 again). Therefore, we cannot have more than four columns in this case, implying that if the dimension, n, of the grid is greater than three, there must be at least $3n + 1$ cells.

But we can't have only $3n + 1$ cells either. If we did, then exactly one row would have four cells. Because the puzzle has rotational symmetry, a single row with four cells can only occur as the middle row

Figure 2. Obstructions arising from using only $3n$ cells.



in an odd-dimension grid. However, the symmetry structure rule then implies that these four cells must sit in the center of that row, which is impossible due to the odd length. As such, there must be at least $3n + 2$ cells in an $n \times n$ puzzle with $n > 3$.

Realizing the Lower Bound on Cells

Having shown that the number of cells is at least $3n + 2$, we now exhibit a general puzzle pattern for an $n \times n$ puzzle that contains exactly $3n + 2$ cells (and thus exactly $n^2 - 3n - 2$ voids).

Number the rows and columns for the $n \times n$ crossword puzzle grid from 1 to n starting at the bottom left corner. In column 1, put a cell in rows 1, 2, and 3. For column k (with k ranging from 2 to $n - 1$), place a cell in rows $k - 1$, k, and $k + 1$. Finally, in column n, put a cell in rows $n - 2$, $n - 1$, and n. This process requires $3n$ cells but leaves the first row and the last row with Across answers of length only two. To rectify this, we place one cell in column 3 of row 1 and one cell in column $n - 2$ of row n. The four puzzles in figure 3 serve as representatives that this pattern, which we call the *stairstep pattern,* creates a crossword puzzle grid satisfying all of the structure rules.

Note that the stairstep pattern is not the only way to realize the lower bound of cells for $n \times n$ grids with $n > 8$. We invite the reader to construct an allowable, non-stairstep 9×9 grid with 29 cells and then try to answer Question 5.

Counting Voids

The previous section shows how to count the cells in a stairstep puzzle by proceeding from left to right and adding the necessary extra cell in each of columns 3 and $n - 2$. This provides one method to count the number of voids; however, we can also count the voids directly in a different way. The void pattern in the stairstep puzzle nearly creates two triangular arrays, each representing the sum of the natural numbers from 1 to $n - 2$. Figure 4 shows these arrays by adding two gray squares to the 11×11 stairstep puzzle voids (pictured in red and blue). Thus, we get a total of

$$2 \cdot \left[\left(\sum_{i=1}^{n-2} i\right) - 2\right]$$

voids in the stairstep pattern. The sum in the formula represents a classic first exposure to induction and has a closed form given by $(n-2)(n-1)/2$. Thus, the maximum number of voids in an $n \times n$ puzzle is

$$\begin{aligned} 2 \cdot (n-2)(n-1)/2 - 4 &= n^2 - 3n + 2 - 4 \\ &= n^2 - 3n - 2, \end{aligned}$$

which matches with our determination that there are $3n + 2$ cells in such a puzzle.

Figure 3. The stairstep puzzles for dimensions four through seven. The orange shaded cells are the two added cells.

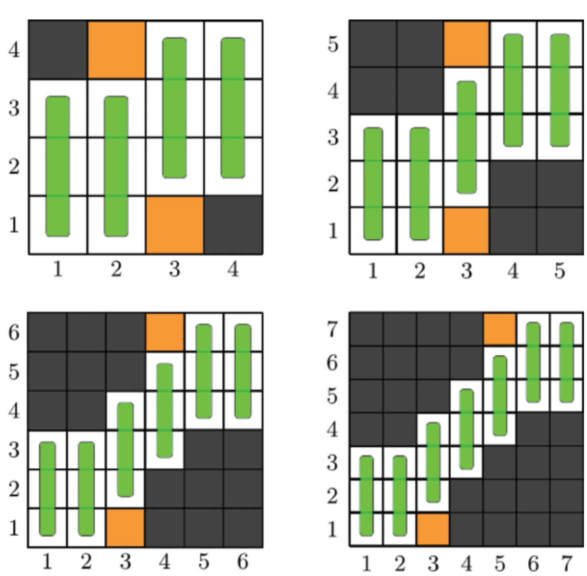
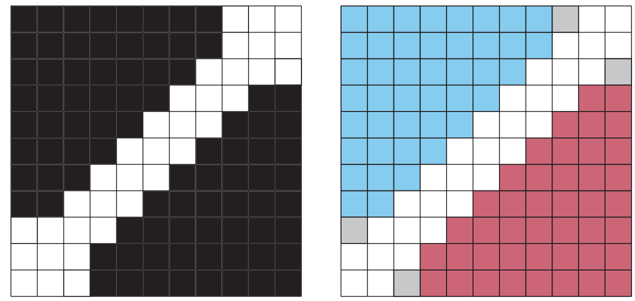


Figure 4. The 11 stairstep pattern converted to two almost-triangular arrays.



Puzzling Patterns

The field of crossword puzzle mathematics seems to be relatively unexplored; there are numerous other questions to investigate. The next time you see a crossword puzzle, in addition to finding the solution, ask yourself what mathematical questions you might also be able to answer!

This work was completed by four math majors at Western Oregon University as part of their senior capstone experience in the 2021–2022 academic year: ***Angel Cruz, Brandilan Moring, TJ Rabosky,*** *and* ***Haley Willmott,*** *along with their professors* ***Dr. Ben Coté*** *and* ***Dr. Leanne Merrill****. The former students are now proud Western Oregon alumni with mathematics degrees who use mathematics in creative ways in their various careers.*

No potential conflict of interest was reported by the author.

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