MTH 410 - Homework 3 - Due: 6/8/2016

You must complete 8 of the following problems to get any credit. If you do more than 8 you will get extra credit. The more that you do the more credit you get.

1. (Counts as two problems) Let (X, τ_X) and (Y, τ_Y) be topological spaces. Let $f : X \to Y$ be a continuous function. Define a relation $\sim$ on X by

   - For any $a, b \in X$, $a \sim b$ if and only if $f(a) = f(b)$.

   (a) Prove that $\sim$ is an equivalence relation on X.

   (b) Define a function
   $$F : X/\sim \longrightarrow Y,$$
   by $F([x]) = f(x)$. Prove the following statements about F.

      i. (Well-defined) For any $x \in X$ and any $y \in [x]$, we have that $F([x]) = F([yx)$.

      ii. (Continuous) The map F is a continuous map.

      iii. (Identification) For any $x \in X$, we have that $(F \circ q_\sim)(x) = f(x)$.

      iv. (Unique) If
      $$G : X/\sim \longrightarrow Z$$
      is any function that satisfies $(G \circ q_\sim)(x) = f(x)$ for all $x \in X$, then $G([x]) = F([x])$ for all $[x] \in X/\sim$.

2. For the function $\exp : \mathbb{R} \to S^1$ defined by $\exp(r) = (\cos(r), \sin(r))$. In our last homework we proved that this is a continuous function. Referring to the previous problem, describe in this case, the space $X/\sim$, the map $q_\sim : X \to X/\sim$, and the map $F : X/\sim \to S^1$.

3. Let (X, d) be a metric space. Define
   $$D : (X \times X) \times (X \times X) \to \mathbb{R}$$
   by
   $$D((x_1, y_1), (x_2, y_2)) = \sqrt{d(x_1, x_2)^2 + d(y_1, y_2)^2}.$$
   You may assume and use without proof that D defines a metric on $X \times X$. Prove that the metric
   $$d : X \times X \to \mathbb{R}$$
   on X is continuous.

4. For topological spaces (X, τ_X) and (Y, τ_Y) a continuous function
   $$f : X \longrightarrow Y$$
   is called an open map if for any subset $U \subset X$, we have that: if $U \in \tau_X$ then $f(U) = \{f(x) | x \in U\} \in \tau_Y$. Consider the projection map $\pi_1 : X \times Y \to X$ defined by $\pi_1(x, y) = x$. Prove that π_1 is an open map.

5. Find a function $f : \mathbb{R} \to \mathbb{R}$ that is continuous at precisely no point in $\mathbb{R}$. Prove that the function that you provide has the desired properties.

6. Find a function $f : \mathbb{R} \to \mathbb{R}$ that is continuous at precisely one point in $\mathbb{R}$. Prove that the function that you provide has the desired properties.

7. Let X and Y be sets and $f : X \to Y$ be a function. Let $U, V \subset Y$. Prove:
   (a) $f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V)$.
   (b) $f^{-1}(U \cap V) = f^{-1}(U) \cap f^{-1}(V)$.
   (c) $f^{-1}(U - V) = f^{-1}(U) - f^{-1}(V)$.

8. Let X and Y be sets and $f : X \to Y$ be a function. Let $U, V \subset X$. For each of the following provide examples of X, Y, U, V and f that show that they are not true
   (a) $f(U \cap V) = f(U) \cap f(V)$.
   (b) $f(U - V) = f(U) - f(V)$.

9. Provide an example of topological spaces X and Y and continuous functions $f : X \to Y$, $g : Y \to X$, such that for all $x \in X$ we have that $g(f(x)) = x$, but there is a $y \in Y$ such that $f(g(y)) \neq y$.

10. Provide an example of topological spaces X and Y and continuous functions $f : X \to Y$, $g : Y \to X$, such that for all $y \in Y$ we have that $f(g(y)) = y$, but there is an $x \in X$ such that $g(f(x)) \neq x$.

11. Let $+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defied by $+(x, y) = x + y$ for all $x, y \in \mathbb{R}$. Prove that $+$ is a continuous map.

12. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Fix $x \in \mathbb{R}$. Define a sequence $x_0 = x$, $x_1 = f(x)$, $x_2 = f(f(x)) = f^2(x)$, and so on, so that $x_n = f^n(x)$, for each $n \geq 0$. Assume that there is a $y \in \mathbb{R}$ such that the sequence x_n converges to y. Prove that $f(y) = y$.

13. Let $f : [0, 1] \to [0, 1]$ be a continuous function. You may assume without proving the fact that any sequence in $[0, 1]$ has a convergent subsequence. Prove that there is a point $y \in \mathbb{R}$ such that $f(y) = y$.

14. Prove that any sequence in $[0, 1]$ has a convergent subsequence.

15. Let p be a polynomial in one variable with real coefficients. Fix $\epsilon > 0$, assume that $p(x) = 0$ for all $x \in (-\epsilon, \epsilon)$. Use the induction and the definition of the derivative to prove that $p(x) = 0$ for all $x \in \mathbb{R}$.