MTH 410 - Homework 2 - Due: 5/9/2016

1. Let $R = \mathbb{R}[x, y]$ be the polynomial ring in two variables with real coefficients. For any $S \subset R$ define the subset $V(S) \subset \mathbb{R}^2$ to be

$$V(S) = \{x \in \mathbb{R}^2 | f(x) = 0 \text{ for all } f \in S\}.$$

A subset $S \subset R$ is an ideal if for any $s \in S$ and any $r \in R$ the product of the two polynomials $s \cdot r \in S$. Let $U \subset \mathbb{R}^2$. Define U to be open if there is an ideal $S \subset R$ such that $\mathbb{R}^2 - U = V(S)$.

You may use the following statements without proof:

- For any collection $\{S_\alpha\}_{\alpha \in \Lambda}$ of ideals in R, there exists and ideal, ΣS_α, such that

$$\bigcap_{\alpha \in \Lambda} V(S_\alpha) = V(\Sigma S_\alpha).$$

- For any finite collection $S_1, S_2, \cdots S_n$ of ideals in R, there is an ideal $\cap S_i$ such that

$$\bigcup_{i=1}^{n} V(S_i) = V(\cap S_i).$$

(a) Prove that the collection of all such open sets forms a topology on $\mathbb{R}^2$.

(b) Is this the same as the standard topology on $\mathbb{R}^2$? Prove it, or provide a counterexample of a set that is open in one topology but not in the other.

2. Let $S^1 = \{(x, y) \in \mathbb{R}^2 | \sqrt{x^2 + y^2} = 1\}$. Define S^1 to have the subspace topology with respect to the standard topology on $\mathbb{R}^2$. Define the map $\exp : \mathbb{R} \to S^1$ by

$$\exp(r) = (\cos(r), \sin(r))$$

for any $r \in \mathbb{R}$.

(a) Prove that $\exp(r)$ is continuous.

(b) What is the exact value of all points in $\exp^{-1}\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$?

For topological spaces X and Y, define the set $\mathcal{C}(X, Y)$ to be the set of all continuous maps from X to Y, and define the set $\mathcal{H}(X, Y)$ to be the set of all homeomorphisms from X to Y. Note that $\mathcal{H}(X, Y) \subset \mathcal{C}(X, Y)$.

A **monoid** is a set M endowed with an associative binary operation

$$\cdot : M \times M \to M : (x, y) \mapsto x \cdot y$$

such that there exists an element $e \in M$ satisfying that for all $x \in M$ that $e \cdot x = x \cdot e = x$. Recall that associativity means, for all $x, y, z \in M$, that $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.

A moinoid M is a **group** if for every $x \in M$ there is an element $x^{-1} \in M$ such that $x \cdot x^{-1} = x^{-1} \cdot x = e$.

3. Let $\star$ be the set containing one element endowed with the discrete topology.

   (a) For any topological space X, what is the set $\mathcal{C}(X, \star)$? Justify your answer.

   (b) For any topological space X, what is the set $\mathcal{C}(\star, X)$? Justify your answer.

4. (a) For any topological space X prove that the set $\mathcal{C}(X, X)$ is a monoid under the operation of function composition $f \cdot g = f \circ g$, where $(f \circ g)(x) = f(g(x))$, for all $x \in X$.

   (b) Prove for any topological space X that the set $\mathcal{H}(X, X)$ is a group under the operation of function composition $f \cdot g = f \circ g$.