MTH 410 - Homework 1 - Due: 4/22/2016

1. Let X be the collection of all continuous function from the closed unit interval $[0,1]$ to the real numbers $\mathbb{R}$. Define the function

$$d(f,g) = \sup\{|f(x) - g(x)| \text{ for } x \in [0,1]\}.$$

   (a) Prove that d is a metric on X.

   (b) Is d a metric if the word continuous is is removed from the definition of X? Justify you answer.

   (c) Find the exact value of $d(\sqrt{x}, x^2)$. Show your work.

Proof: We verify the four conditions required to be a metric:

- $d(f,g) \geq 0$ for every $f, g \in X$,
  Let $f, g \in X$, since $d(f,g)$ is defined to be a supremum we know that $d(f,g) \geq |f(x) - g(x)|$ for all $x \in [0,1]$. The absolute value function takes only non-negative values, so we know that $|f(x) - g(x)| \geq 0$ for all $x \in [0,1]$. Thus for any $x \in [0,1]$ we have that $d(f,g) \geq |f(x) - g(x)| \geq 0$ as desired.
- For any $f, g \in X$, $d(f,g) = 0$ if and only if $f = g$,
  Let $f, g \in X$, assume that $f = g$. This means that $f(x) = g(x)$ for all $x \in [0,1]$, thus $|f(x) - g(x)| = 0$ for all $x \in [0,1]$. Therefore, we have that $d(f,g) = \sup\{0\} = 0$. Now assume that $d(f,g) = 0$. This means that $0 = d(f,g) \geq |f(x) - g(x)| \geq 0$ for all $x \in [0,1]$. Thus $|f(x) - g(x)| = 0$ for all $x \in [0,1]$. This implies that $f(x) = g(x)$ for all $x \in [0,1]$, and thus $f = g$.
- $d(f,g) = d(g,f)$, for every $f, g \in X$,
  Let $f, g \in X$. Since $|f(x) - g(x)| = |(-1)(g(x) - f(x))| = |g(x) - f(x)|$ we have the equality of sets $\{|f(x) - g(x)| : x \in [0,1]\} = \{|g(x) - f(x)| : x \in [0,1]\}$. Thus the supremum of each of these two sets is equal. This means that $d(f,g) = d(g,f)$.
- $d(f,h) \leq d(f,g) + d(g,h)$, for all $f, g, h \, inX$.
  Let $f, g, h \in X$. For any $x \in [0,1]$ the definition of supremum and the triangle inequality for absolute value give us the following:

$$|f(x) - h(x)| = |f(x) - g(x) + g(x) - h(x)| \leq |f(x) - g(x)| + |g(x) - h(x)| \leq d(f,g) + d(g,h).$$

  This implies that $d(f,g) + d(g,h)$ is an upper bound for the set $\{|f(x) - h(x)| : x \in [0,1]\}$. Thus by the definition of supremum

$$d(f,h) \leq d(f,g) + d(g,h).$$

  Therefore d is a metric on X•

2. For $x = (x_1, x_2)$, and $y = (y_1, y_2)$ in $\mathbb{R}^2$ define

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$$

(a) Prove that d is a metric on $\mathbb{R}^2$.

Proof: We verify the four conditions required to be a metric:

- $d(x, y) \geq 0$ for every $x, y \in \mathbb{R}^2$,
  The output values of the function $f(x) = \sqrt{x}$ are always non-negative. Therefore this property is satisfied.
- For any $x, y \in \mathbb{R}$, $d(x, y) = 0$ if and only if $x = y$,
  Let $x = (x_1, x_2), y = (y_1, y_2)$ for $x_1, x_2, y_1, y_2 \in \mathbb{R}$. Note that $f(z) = \sqrt{z} = 0$ if and only if $z = 0$. Thus $d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} = 0$ if and only if $(x_1 - y_1)^2 + (x_2 - y_2)^2 = 0$. since each summand is non-negative, this equality is true if and only if both $(x_1 - y_1)^2 = 0$ and $(x_2 - y_2)^2 = 0$. The first of these is ture if and only if $x_1 = y_1$, and the second is true if and only if $x_2 = y_2$. These last statements are true if and only if $x = y$. This is the desired result.
- $d(x, y) = d(y, x)$, for every $x, y \in \mathbb{R}$,
  Let $x = (x_1, x_2), y = (y_1, y_2)$ for $x_1, x_2, y_1, y_2 \in \mathbb{R}$. Since $(x_1 - y_1)^2 = (y_1 - x_1)^2$ and $(x_2 - y_2)^2 = (y_2 - x_2)^2$ we have that

  $$d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} = \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2} = d(y, x).$$

- $d(x, z) \leq d(x, y) + d(y, z)$, for all $x, y, z \in \mathbb{R}$.
  Let $x = (x_1, x_2), y = (y_1, y_2), z = (z_1, z_2)$ for $x_1, x_2, y_1, y_2, z_1, z_2 \in \mathbb{R}$. First define $a = x_1 - y_1, b = y_1 - z_1, c = x_2 - y_2, d = y_2 - z_2$. Now note that $d(x, y)^2 = a^2 + c^2$ and $d(y, z)^2 = b^2 + d^2$, and

  $$d(x, z)^2 = (x_1 - z_1)^2 + (x_2 - z_2)^2 = (x_1 - y_1 + y_1 - z_1)^2 + (x_2 - y_2 + y_2 - z_2)^2 = (a+b)^2 + (c+d)^2.$$

  Expanding this last equation and simplifying using the two equations before it gives

  $$d(x, z)^2 = a^2 + 2ab + b^2 + c^2 + 2cd + d^2 = d(x, y)^2 + d(y, z)^2 + 2(ac + bd).$$

  The second thing to note is that

  $$(d(x, y) + d(y, z))^2 = d(x, y)^2 + 2d(x, y)d(y, z) + d(y, z)^2.$$

  Lemma:

  $$ab + cd \leq d(x, y)d(y, z).$$

  Proof of Lemma First note that

  $$0 \leq (ac - bd)^2 = a^2c^2 - 2adbc + b^2d^2,$$

  and therefore we have that $2adbc \leq a^2c^2 + b^2d^2$. Adding $a^2b^2 + c^2d^2$ to both sides gives

  $$a^2b^2 + 2adbc + c^2d^2 \leq a^2b^2 + a^2c^2 + b^2d^2 + c^2d^2$$

  Now note that the left side is $(ab + cd)^2$, and the right side is

  $$a^2b^2 + a^2c^2 + b^2d^2 + c^2d^2 = (a^2 + c^2)(b^2 + d^2) = d(x, y)^2 d(y, z)^2$$

Thus we have proven that

$$(ab+cd)^2 \leq (d(x,y)d(y,z))^2.$$

Again since the function $z \mapsto z^2$ is an order preserving function on the positive real numbers, this proves the desired inequality$\bullet$

With this inequality, we have proven that

$$d(x,z)^2 \leq (d(x,y)+d(y,z))^2.$$

Since the function $z \mapsto z^2$ is an increasing function on the positive real numbers, and therefore order preserving there, we have that

$$d(x,z) \leq d(x,y)+d(y,z),$$

and therefore that d satisfies the triangle inequality.

Therefore d is a metric on $\mathbb{R}$$\bullet$

(b) Define a sequence in $\mathbb{R}^2$ by $x_n = \left(\frac{1}{n}\cos(n), \frac{1}{n}\sin(n)\right)$, for $n \in \mathbb{N}$. Decide whether or not this sequence converges. Either prove that it does not converge, or prove that it does and find its limit point.

Claim: This sequence converges to $0 = (0,0)$.

Proof: Let $\epsilon > 0$, and choose N such that $1/N \leq \epsilon$. Then for any $n \geq N$, we have that $1/n \leq 1/N < \epsilon$. Thus we have

$$d(x_n, 0) = \sqrt{(\frac{1}{n}\cos(n)-0)^2 + (\frac{1}{n}\sin(n)-0)^2} = \sqrt{\frac{1}{n^2}(\cos^2(n)+\sin^2(n))} = \sqrt{\frac{1}{n^2}} = \frac{1}{n} < \epsilon.$$

This proves that this sequence converges to 0$\bullet$

3. Let (X, d_X) and (Y, d_Y) be metric spaces, and $f : X \to Y$ be a continuous function.

   (a) Prove that if the sequence $\{x_n\}_{n=1}^{\infty}$ in X converges to $x \in X$, then the sequence $\{f(x_n)\}_{n=1}^{\infty}$ in Y converges to $f(x) \in Y$.

   Proof: Let $\epsilon > 0$, then since f is continuous there exists a $\delta > 0$ such that for all $z \in X$ if $d(z, x) < \delta$, then $d(f(z), f(x)) < \epsilon$. Since the sequence $\{x_n\}_{n=1}^{\infty}$ converges to x, there exists, for this choice of δ an $N \geq 1$ such that for all $n \geq N$ we have that $d(x_n, x) < \delta$. But by the previous sentence, this implies that $d(f(x_n), f(x)) <\leq \epsilon$. This proves that the sequence $\{f(x_n)\}_{n=1}^{\infty}$ converges to $f(x)$$\bullet$

   (b) Provide an example of a continuous function $f : \mathbb{R} \to \mathbb{R}$, and a sequence $\{x_n\}_{n=1}^{\infty}$ in $\mathbb{R}$ such that the sequence $\{f(x_n)\}_{n=1}^{\infty}$ converges, but the sequence $\{x_n\}_{n=1}^{\infty}$ does not converge.

   Example: For each $n \geq 1$ define $x_n = (-1)^n$. The sequence does not converge since for any $N \geq 0$ there is an $n \geq N$ such that $1 = d(x_n, x_{n+1}) \leq d(x_n, x) + d(x, x_{n+1})$, for every $x \in \mathbb{R}$. Let $f(x) = |x|$, then for every open interval $(a, b) \subset \mathbb{R}$, we have that $f^{-1}(a, b) = (-b, -a) \cup (a, b)$ is the union of two open intervals, and thus is open. Since open intervals form a basis for the topology on $\mathbb{R}$ we have that f is continuous. Note that $f(x_n) = 1$ for all n, so that the sequence $\{f(x_n)\}_{n=1}^{\infty}$ is a constant sequence. Thus for any $\epsilon > 0$ let $N = 1$, then for any $n \geq N$ we have that $d(f(x_1), 1) = 0 < \epsilon$. Thus this sequence converges$\bullet$

4. Let X and Y be sets a function $f : X \to Y$ is called a bijection if there exists a function $g : Y \to X$ such that both of the following equalities hold

$$g(f(x)) = x \text{ for all } x \in X,$$

and

$$f(g(y)) = y \text{ for all } y \in Y.$$

Prove that such a function g is unique. That is to say that if there are function $g_1 : Y \to X$ and $g_2 : Y \to X$ each of which satisfys the above equations, then $g_1(y) = g_2(y)$, for all $y \in Y$.

Proof: Assume that we have functions $g_1, g_2 : Y \to X$ such that $g_1(f(x)) = x = g_2(f(x))$ for all $x \in X$, and $f(g_2(y)) = y = f(g_2(y))$ for all $y \in Y$. Let $y \in Y$, define $z_1 = g_1(y)$, and $z_2 = g_2(y)$. By using these identities we have the following

$$g_1(y) = g_1(f(g_2(y)) = g_1(f(z_2)) = z_2 = g_2(y).$$

Thus $g_1 = g_2\bullet$