MTH 410 - Homework 1 - Due: 4/22/2016

1. Let X be the collection of all continuous function from the closed unit interval $[0,1]$ to the real numbers $\mathbb{R}$. Define the function

$$d(f,g) = \sup\{|f(x) - g(x)| \text{ for } x \in X\}.$$

   (a) Prove that d is a metric on X.

   (b) Is d a metric if the word continuous is is removed from the definition of X? Justify you answer.

   (c) Find the exact value of $d(\sqrt{x}, x^2)$. Show your work.

2. For $x = (x_1, x_2)$, and $y = (y_1, y_2)$ in $\mathbb{R}^2$ define

$$d(x,y) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

   (a) Prove that d is a metric on $\mathbb{R}^2$.

   (b) Define a sequence in $\mathbb{R}^2$ by $x_n = \left(\frac{1}{n}\cos(n), \frac{1}{n}\sin(n)\right)$, for $n \in \mathbb{N}$. Decide whether or not this sequence converges. Either prove that it does not converge, or prove that it does and find its limit point.

3. Let (X, d_X) and (Y, d_Y) be metric spaces, and $f : X \to Y$ be a continuous function.

   (a) Prove that if the sequence $\{x_n\}_{n=1}^{\infty}$ in X converges to $x \in X$, then the sequence $\{f(x_n)\}_{n=1}^{\infty}$ in Y converges to $f(x) \in Y$.

   (b) Provide an example of a continuous function $f : \mathbb{R} \to \mathbb{R}$, and a sequence $\{x_n\}_{n=1}^{\infty}$ in $\mathbb{R}$ such that the sequence $\{f(x_n)\}_{n=1}^{\infty}$ converges, but the sequence $\{x_n\}_{n=1}^{\infty}$ does not converge.

4. Let X and Y be sets a function $f : X \to Y$ is called a bijection if there exists a function $g : Y \to X$ such that both of the following equalities hold

$$g(f(x)) = x \text{ for all } x \in X,$$

   and

$$f(g(y)) = y \text{ for all } y \in Y.$$

   Prove that such a function g is unique. That is to say that if there are function $g_1 : Y \to X$ and $g_2 : Y \to X$ each of which satisfys the above equations, then $g_1(y) = g_2(y)$, for all $y \in Y$.