INTERSECTION CUP AND CAP PRODUCTS

BRYCE CHRIESTENSON

ABSTRACT. We define the intersection homology and cohomology for an abstract perversity p on a stratified pseudo-manifold. We then define an intersection cup and cap product, which mimic the constructions for singular homology and cohomology.

1. DEFINITIONS

Definition 1.1. An n-dimensional stratified pseudo-manifold X is a paracompact Hausdorff space together with a filtration by closed subsets

$$\emptyset = X^{-1} \subseteq X^0 \subseteq \cdots \subseteq X^{n-2} \subseteq X^{n-1} \subseteq X^n = X$$

such that

- $X \setminus X^{n-1}$ is dense in X, and
- for each $x \in X^i \setminus X^{i-1}$, there is a neighborhood U of x for which there is a compact $n - i - 1$ dimensional stratified pseudo-manifold L and a homeomorphism
$$\phi : \mathbb{R}^i \times cL \to U$$
that takes $R^i \times C\left(L^{j-1}\right)$ into $X^{i+j} \cap U$. A neighborhood U with this property is called distinguished and L is called the link of x.

The X^i are called the i-skeleta. Write $X_i = X^i \setminus X^{i-1}$; this is a topological i-manifold that may be empty. The connected components of X_i are called the i-strata, denote them by $\mathcal{S}_X^i = \{S \subset X_i | \ S \text{ is a connected component of } X_i\}$. Thus the collection of all strata of X is $\mathcal{S}_X = \cup_i \mathcal{S}_X^i$. The elements of $\mathcal{S}_X^n$ are called the regular strata. All other strata are called singular strata. The co-dimension of a stratum $S \in \mathcal{S}_X^i$ is defined $\operatorname{codim}(S) = n - i$.

Definition 1.2. A general perversity on X is a function $\overline{p} : \mathcal{S}_X \to \mathbb{Z}$ such that $\overline{p}|_{\mathcal{S}_X^0} = 0$. That is to say $\overline{p}(S) = 0$ whenever $\operatorname{codim}(S) = 0$. The top perversity is $\overline{t}(S) = \operatorname{codim}(S) - 2$, whenever $\operatorname{codim}(S) \neq 0$, and $\overline{t}(S) = 0$ when $\operatorname{codim}(S) = 0$. Given $\overline{p}$, define the dual perversity $D\overline{p}(S) = \overline{t}(S) - \overline{p}(S)$.

Remark 1.3. Recall, for a topological space X, and an abelian group G, the collection of singular i-simplicies, $C_i(X; G)$, in X, with coefficients in G, is the free abelian group on the collection of all continuous maps $\sigma : \Delta^i \to X$, where Δ^i is the standard i-simplex. The collection $\{C_i(X; G)\}_{i \in \mathbb{N}}$ forms a simplicial set. Whose face and degeneracy maps are defined by pulling simplicies back along the face and degeneracy maps defined on the standard simplicies Δ^*.

For the simplex Δ^i the k-skeleton, $k \leq i$, is the collection of k-dimensional sub-faces of Δ^i, and is denoted Δ_k^i.

Unless it is stated otherwise we will assume that $G = F$ is a field of characteristic zero.

Definition 1.4. Given an n-dimensional stratified pseudo-manifold X, and a general perversity $\overline{p}$ a singular i-simplex $\sigma : \Delta^i \to X$ in $C_i(X; G)$ is $\overline{p}$-allowable if

$$\sigma^{-1}(S) \subset \Delta^i_{i - \operatorname{codim}(S) + \overline{p}(S)},$$

for every $S \in \mathcal{S}_X$.

A chain $\xi \in C_i(X;G)$ is $\overline{p}$-allowable if every simplex in ξ and $d\xi$ is $\overline{p}$-allowable. The collection $\overline{p}$-allowable chains is a subcomplex

$$I^{\overline{p}}C_\bullet(X;G) \subset C_\bullet(X;G),$$

called the intersection chain complex. Its homology $I^{\overline{p}}H_\bullet(X;G)$ is called the $\overline{p}$-intersection homology.

Remark 1.5. For perversities $\overline{p}, \overline{q}$ on X, notice that if $\overline{p}(S) \geq \overline{q}(S)$, for all $S \in \mathcal{S}_X$, then we have that $I^{\overline{p}}C_\bullet(X;G)$ is a subcomplex of $I^{\overline{q}}C_\bullet(X;G)$.

2. Products of pseudo-manifolds, and the IH Künneth theorem

Remark 2.1. Let X and Y be stratified pseudo-manifolds of dimension m and n respectively. For strata $S \in \mathcal{S}_X$ and $Z \in \mathcal{S}_Y$ define the stratum $S \times Z \in \mathcal{S}_{X\times Y}$. This defines a stratification on $X \times Y$ which makes a it into a stratified pseudo-manifold, by taking as the l-skeleton, the union of all strata of dimension less than or equal to l. Notice that the dimension of $S \times Z$ is the sum of the dimension of S and the dimension of Z. Thus $\mathrm{codim}(S \times Z) = \mathrm{codim}S + \mathrm{codim}Z$.

Definition 2.2. Let X, Y be a stratified pseudo-manifolds. Let $S \in \mathcal{S}_X$, and $Z \in \mathcal{S}_Y$. Define the perversity $\delta_{X\times Y}(S \times Z) = 2$, if both $\mathrm{codim}S \neq 0$, and $\mathrm{codim}Z \neq 0$, and $\delta_{X\times Y}(S \times Z) = 0$, otherwise.

Given perversities $\overline{p}$ on X, and $\overline{q}$ on Y define their sum on $X \times Y$ to be $\overline{p+q}(S \times Z) = \overline{p}(S) + \overline{q}(Z)$.

Define the perversity

$$Q_{\overline{p},\overline{q}}(S \times Z) = \overline{p+q}(S \times Z) + \delta_{X\times Y}(S \times Z).$$

Remark 2.3. Suppose that $\sigma : \Delta^i \to X$, a $\overline{p}$-admissible i-simplex, and $\tau : \Delta^j \to Y$ an $\overline{q}$-admissible j-simplex. Note that the cross product $\sigma \times \tau : \Delta^i \times \Delta^j \to X \times Y$ satisfies $(\sigma \times \tau)^{-1}(S \times Z)$ is contained in the $m+n-\mathrm{codim}(S \times Z)+\overline{p+q}(S \times Z)$-skeleton of $\Delta^i \times \Delta^j$,

$$\left(\Delta^i \times \Delta^j\right)_{m+n-\mathrm{codim}(S\times Z)+\overline{p+q}(S\times Z)}.$$

One can decompose $\Delta^i \times \Delta^j$ into $(i+j)$-simplicies with non-overlapping interiors, via maps

$$EZ_\alpha : \Delta^{i+j} \to \Delta^i \times \Delta^j,$$

for some collection of α, in such a way that

$$EZ_\alpha^{-1}\left(\left(\Delta^i \times \Delta^j\right)_l\right) = \Delta_l^{i+j}.$$

Thus we have that $(\sigma \times \tau) \circ EZ_\alpha : \Delta^{i+j} \to X \times Y$ is $(\overline{p+q})$-admissible.

The Eilenberg-Zilber morphism $EZ : C_\bullet(X;G) \otimes C_\bullet(Y;G) \to C_\bullet(X \times Y;G)$, is defined on simplicies by

$$EZ(\sigma \times \tau) = \sum_\alpha (-1)^{n_\alpha}(\sigma \times \tau) \circ EZ_\alpha.$$

The Künneth theorem states that EZ is a quasi-isomorphism of chain complexes. We see that we now have the following result

Proposition 2.4. *The Eilenberg-Zilber map for singular homology restricts to give a map*

$$I^{\overline{p}}C_\bullet(X;G) \otimes I^{\overline{q}}C_\bullet(Y;G) \to I^Q C_\bullet(X \times Y;G),$$

where Q is any perversity that satisfying $Q(S \times Z) \geq \overline{p+q}(S \times Z)$ for all $S \in \mathcal{S}_X$, and $Z \in \mathcal{S}_Y$.

Theorem 2.5 (IH Künneth Theorem)**.** *In the case that $Q = Q_{\overline{p},\overline{q}}$ the Eilenberg-Zilber map induces an isomorphism*

$$EZ_* : I^{\overline{p}}H_\bullet\left(X;G\right) \otimes I^{\overline{q}}H_\bullet\left(Y;G\right) \to I^{Q_{\overline{p},\overline{q}}}H_\bullet\left(X \times Y;G\right).$$

Remark 2.6. It should be noted that here it is necessary for G to be a field. Furthermore the Eilenberg-Zilber map EZ is natural in both X and Y, and associative up to isomorphism.

By using this isomorphism, along with the stratification induced on an open set $U \subset X$, $V \subset Y$, we apply the five lemma to the pairs (X, A), and (Y, B), to get the relative version of the Künneth theorem

Theorem 2.7 (Relative IH Künneth Theorem)**.** *In the case that $Q = Q_{\overline{p},\overline{q}}$, and $U \subset X$, $V \subset Y$, are open, the Eilenberg-Zilber map induces an isomorphism*

$$EZ_* : I^{\overline{p}}H_\bullet\left(X,A;G\right) \otimes I^{\overline{q}}H_\bullet\left(Y,B;G\right) \to I^{Q_{\overline{p},\overline{q}}}H_\bullet\left(X \times Y, \left(A \times Y\right) \cup \left(X \times B\right);G\right).$$

3. The diagonal map, and perverse coalgebra structure

Remark 3.1. We consider the case when $X = Y$, in this case we have the diagonal map $\nabla : X \to X \times X : x \mapsto (x, x)$. This gives a chain complex morphism $\nabla_* : C_\bullet\left(X;G\right) \to C_\bullet\left(X \times X;G\right)$. Let $\sigma : \Delta_i \to X$ be a $\overline{p}$-admissible i-simplex. Then $\nabla_*\sigma$ is an i-simplex $\nabla_*\sigma = \nabla \circ \sigma : \Delta^i \to X \times X$.

The only strata of $X \times X$ that intersect the image of $\nabla \circ \sigma$ are of the form $S \times S$ for $S \in \mathcal{S}_X$. We know that $\sigma^{-1} \circ \nabla^{-1}\left(S \times S\right) = \sigma^{-1}\left(S\right)$.

We would like to use the above Künneth theorem combined with this diagonal map to define a perverse coproduct

$$I^{\overline{r}}H_\bullet\left(X;G\right) \to I^{\overline{p}}H_\bullet\left(X;G\right) \otimes I^{\overline{q}}H_\bullet\left(X;G\right),$$

for some perversities $\overline{p}, \overline{q}$, and $\overline{r}$. This being the goal we want the above diagonal map to take $\overline{r}$-admissible simplicies to $Q_{\overline{p},\overline{q}}$-admissible simplicies. Then $EZ^{-1} \circ \nabla_*$ will be the desired perverse coproduct.

Thus we want $\nabla_*\sigma$ to be $Q_{\overline{p},\overline{q}}$-admissible, which is equivalent to

$$\sigma^{-1}\left(S\right) \subset \Delta^i_{i-\mathrm{codim}(S \times S) - Q_{\overline{p},\overline{q}}}.$$

So, if σ is $\overline{r}$-admissible then we must have the following relation

$$i - \mathrm{codim}S + \overline{r}\left(S\right) \leq i - 2\mathrm{codim}S + \overline{p}\left(S\right) + \overline{q}\left(S\right) + \delta_{X \times X}\left(S \times S\right).$$

By, simplifying, and rewriting both sides in terms of D, we see this is equivalent to the condition

$$D\overline{t} \geq D\overline{p} + D\overline{q}.$$

Define the notation $Q_{\overline{p},\overline{q},\overline{r}} = Q_{Q_{\overline{p},\overline{q}},\overline{r}} = Q_{\overline{p},Q_{\overline{q},\overline{r}}}$. Using this notation we get the following statement

Proposition 3.2. *If $D\overline{r}\left(S\right) \geq D\overline{p}\left(S\right) + D\overline{q}\left(S\right)$ for each $S \in \mathcal{S}_X$, then ∇ induces a map*

$$\nabla_* : I^{\overline{r}}C_\bullet\left(X;G\right) \to I^{Q_{\overline{p},\overline{q}}}C_\bullet\left(X \times X;G\right).$$

Remark 3.3. Similarly, one could define $\nabla_* \times 1$ and $1 \times \nabla_*$, with the appropriate adjustments to the domain and codomain. One should note also that the map ∇_* is natural in X, and coassociative up to isomorphism.

Definition 3.4. If $D\overline{r} \geq D\overline{p} + D\overline{q}$ then define

$$\Delta = EZ^{-1} \circ \nabla_* : I^{\overline{r}}C_\bullet\left(X;G\right) \to I^{\overline{p}}C_\bullet\left(X;G\right) \otimes I^{\overline{q}}C_\bullet\left(X;G\right).$$

Proposition 3.5. *The map Δ is coassociative. That is to say: If $D\overline{s} \geq D\overline{u} + D\overline{r}$, $D\overline{s} \geq D\overline{p} + D\overline{v}$, $D\overline{u} \geq D\overline{p} + D\overline{q}$, and $D\overline{v} \geq D\overline{q} + D\overline{r}$, then the following diagram commutes*

$$\begin{tikzcd}
I^{\overline{s}}H_\bullet(X;G) \arrow[r, "\Delta"] \arrow[d, "\Delta"] & I^{\overline{u}}H_\bullet(X;G) \otimes I^{\overline{r}}H_\bullet(X;G) \arrow[d, "\Delta\otimes 1"] \\
I^{\overline{p}}H_\bullet(X;G) \otimes I^{\overline{v}}H_\bullet(X;G) \arrow[r, "1\otimes\Delta"] & I^{\overline{p}}H_\bullet(X;G) \otimes I^{\overline{q}}H_\bullet(X;G) \otimes I^{\overline{r}}H_\bullet(X;G)
\end{tikzcd}$$

Proof. This follows from the associative and naturality of the maps ∇_* and EZ. $\square$

Similarly we have that Δ is cocommutative and counital, where the augmentation is given by $\epsilon : I^{\overline{p}}H_\bullet(X;G) \to G$ is zero on $I^{\overline{p}}H_k(X;G)$, when $k > 0$, and $\epsilon\left(\sum_i g_i p_i\right) = \sum_i g_i$ for zero chains. This satisfies $(\epsilon \otimes 1) \circ \Delta = id_{I^{\overline{p}}H_\bullet(S;G)}$. Thus we have

Proposition 3.6. *The family of graded vector spaces $I^{\overline{p}}H_\bullet(X;G)$, where $\overline{p}$ is a perversity on X, along with the maps Δ, and ϵ, form a perverse coassociative, cocommutative, counital, coalgebra.*

Remark 3.7. We also have the same results as above but in the relative case. Let $U \subset X$, $V \subset Y$ open, then notice that ∇_* induces a map

$$\nabla_* : C_\bullet(X, A \cup B; G) \to C_\bullet(X \times X, (A \cup X) \times (X \cup B); G).$$

Similar arguments to those given above show that this descend to the intersection chains level. The composition of ∇_* with the relative Künneth isomorphism yields the relative diagonal.

Proposition 3.8. *Let $D\overline{r} \geq D\overline{p} + D\overline{q}$ then we have a map*

$$\Delta : I^{\overline{r}}H_\bullet(X, A \cup B; G) \to I^{\overline{p}}H_\bullet(X, A; G) \otimes I^{\overline{q}}H_\bullet(X, B; G).$$

Remark 3.9. Because of both the cross product and the connecting homomorphism are natural, and the cross product is induced from a chain homomorphism we have

Proposition 3.10. *Let $i : A \to X$ be the inclusion of an open set A of X. Then the diagram commutes*

$$\begin{tikzcd}
I^{\overline{r}}H_\bullet(X, A; G) \arrow[rr, "\Delta"] \arrow[d, "\partial"] & & I^{\overline{p}}H_\bullet(X, A; G) \otimes I^{\overline{q}}H_\bullet(X;G) \arrow[d, "\partial\otimes 1"] \\
I^{\overline{r}}H_\bullet(A;G) \arrow[r, "\Delta"] & I^{\overline{p}}H_\bullet(A;G) \otimes I^{\overline{q}}H_\bullet(A;G) \arrow[r, "1\otimes i"] & I^{\overline{p}}H_\bullet(A;G) \otimes I^{\overline{q}}H_\bullet(X;G)
\end{tikzcd}$$

4. Intersection cochains and the cup product

Definition 4.1. Given a stratified pseudo-manifold X, and a perversity $\overline{p}$ on X, the $\overline{p}$-intersection cochain complex is

$$I_{\overline{p}}C^\bullet(X;G) = \mathrm{Hom}_G\left(I^{\overline{p}}H_\bullet(X;G), G\right),$$

with the boundary morphism induced from that on $I^{\overline{p}}H_\bullet(X;G)$. The cohomology of this complex is called the $\overline{p}$-intersection cohomology and denoted $I_{\overline{p}}H^\bullet(X;G)$.

Remark 4.2. Since we are assuming G is a field we have from the universal coefficient theorem that

$$I_{\overline{p}}H^\bullet(X;G) \cong \mathrm{Hom}_G\left(I^{\overline{p}}H_\bullet(X;A), G\right).$$

We can make the appropriate changes to define the relative cochain complex for which the analogous statement holds true.

Instead of using the usual Koszul sign convention, we use Dold's sign convention, which includes an extra negative sign. That is to say for a cochain α and a chain c we write

$$\delta\alpha\left(c\right) = -\left(-1\right)^{|\alpha|}\alpha\left(\partial c\right).$$

We now defined the cup product on intersection cohomology. It is defined to be dual to the diagonal map on intersection homology.

Definition 4.3. If $D\overline{r} \geq D\overline{p} + D\overline{q}$, then the cap product

$$\smile : I_{\overline{p}}H^{\bullet}\left(X;G\right) \otimes I_{\overline{q}}H^{\bullet}\left(X;G\right) \to I_{\overline{r}}H^{\bullet}\left(X;G\right),$$

is defined by

$$\left(\alpha \smile \beta\right)\left(x\right) = \left(\alpha \otimes \beta\right) \circ \Delta\left(x\right).$$

If one writes $\Delta\left(x\right) = \sum_i y_i \otimes z_i$, then $\left(\alpha \smile \beta\right)\left(x\right) = \sum_i \alpha\left(y_i\right)\beta\left(z_i\right)$.

Remark 4.4. Since $\smile$ is dual to the operator Δ, which is is coassociative, cocommutative, and counital, we have that $\smile$ is associative, commutative, and unital, where the unit $\eta : F \to I^{\overline{p}}H_{\bullet}\left(X;G\right)$ is dual to the augmentation ϵ. Where here we must make the appropriate restrictions on the perversities.

We point out here that the commutativity of the cup product means graded commutativity. That is to say

$$\alpha \smile \beta = \left(-1\right)^{|\alpha||\beta|}\beta \smile \alpha.$$

Notice also, that we can define the relative cup product by using the relative diagonal map. Let $A, B \subset X$ be open and $D\overline{s} \geq D\overline{p} + D\overline{q}$, then we have

$$\smile : I_{\overline{p}}H^{\bullet}\left(X,A;G\right) \otimes I_{\overline{q}}H^{\bullet}\left(X,B;G\right) \to I^{\overline{r}}H_{\bullet}\left(X,A\cup B;G\right).$$

defined in exactly the same way, only using the relative form of Δ.

5. The cap product

Remark 5.1. We have the following diagram defining the cup product of α and β

$$I^{\overline{r}}H_k\left(X,A\cup B;G\right) \xrightarrow{\Delta} \bigoplus_{i+j=k} I^{\overline{p}}H_i\left(X,A;G\right) \otimes_G I^{\overline{q}}H_j\left(X,B;G\right) \xrightarrow{\alpha\otimes\beta} G.$$

Consider replacing α with $id_{I^{\overline{p}}H_{\bullet}(X,A;G)}$, then since $I^{\overline{p}}H_{\bullet}\left(X,A;G\right)$ is a G vector space one replaces G with $I^{\overline{p}}H_{\bullet}\left(X,A;G\right)\otimes G \cong I^{\overline{p}}H_{\bullet}\left(X,A;G\right)$ to get the following diagram, where G is left implicit and $id_{I^{\overline{p}}H_{\bullet}(X,A;G)}$ is denote by 1

$$I^{\overline{r}}H_k\left(X,A\cup B\right) \xrightarrow{\Delta} \bigoplus_{i=0}^{k} I^{\overline{p}}H_{n-i}\left(X,A\right) \otimes I^{\overline{q}}H_i\left(X,B\right) \xrightarrow{1\otimes\beta} I^{\overline{p}}H_{k-|\beta|}\left(X,A\right).$$

This is how the cap product is defined

Definition 5.2. For $D\overline{r} \geq D\overline{p} + D\overline{q}$ define the intersection cap product

$$\frown : I_{\overline{q}}H^i\left(X,B;G\right) \otimes I^{\overline{r}}H_j\left(X,A\cup B;G\right) \to I^{\overline{p}}H_{j-i}\left(X,A;G\right)$$

by

$$\left(\alpha \frown x\right) = \left(1 \otimes \alpha\right) \circ \Delta\left(x\right).$$

Again, if $\Delta\left(x\right) = \sum_i y_i \otimes z_i$, then

$$\left(\alpha \frown x\right) = \sum_k \left(-1\right)^{|\alpha||y_k|}\alpha\left(z_k\right)y_k.$$

The sign comes from the Koszul convention $\left(\alpha \otimes \beta\right)\left(x \otimes y\right) = \left(-1\right)^{|\beta||x|}\alpha\left(x\right)\otimes\beta\left(y\right)$, since β, and x were moved past each other.

Remark 5.3. Let $i : A \to X$ an open inclusion, and $\partial : I^{\overline{p}}H_k\left(X,A;G\right) \to I^{\overline{p}}H_{k-1}\left(A;G\right)$, $\delta : I_{\overline{p}}H^k\left(A;G\right) \to I_{\overline{p}}H^{k+1}\left(X,A;G\right)$ be the connecting homomorphisms. Then based on the work above we have the following list of identities

- $(\alpha \smile \beta) \frown x = \alpha \frown (\beta \frown x)$,
- $\epsilon (\alpha \frown x) = \alpha (x)$,
- $\partial (\alpha \frown x) = (-1)^{|\alpha|} (i^* \alpha) \frown \partial x$,
- $\delta (\alpha) \frown x = (-1)^{|\alpha|} i_* (\alpha \frown \partial x)$,
- $\alpha \frown i_* x = i_* ((i^* \alpha) \frown x)$.

These follow once the correct perversities are chosen, and the correct diagram is shown to commute.

E-mail address: bchriestenson@mathi.uni-heidelberg.de