Equivariant Moore Approximation and Fiber Bundles

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Motivation

The material in this talk comes out of joint work with M. Banagl on intersection spaces and poincaré duality. One of our goals is to attempt to construct a fiberwise-truncated fiber bundle:

Fiberwise k-truncation

The fiberwise k-truncation of the fiber bundle $\pi : E \to B$ consists of a fiber bundle $\pi_{<k} : \mathrm{ft}_{<k}E \to B$, and a morphism of fiber bundles

$$F_{<k} : \mathrm{ft}_{<k}E \to E$$

such that when restricted to a fiber the induced map $f_{<k} : L_{<k} \to L$ is an isomorphism

$$H_r(L_{<k}) \cong H_r(L)$$

when $r < k$, and $H_r(L_{<k}) = 0$ when $r \geq 0$.

Here we assume that both of the above fiber bundles have the same structure group G.

Existential Questions

If one can construct a fiberwise-truncated bundle then there must exist a G-space $L_{<k}$ and a G-equivariant morphism $f_{<k} : L_{<k} \to L$ between the fibers. This is not always the case:

Example

Let G be a group with $H_r(G) \neq 0$ for some $r > 0$ (e.g S^1). Let $\pi : E \to B$ be a principal G-bundle. It is not possible to construct a fiberwise k-truncation of this bundle for any $k \leq r$.

Question

Given a G-space L does there exist a G-space $L_{<k}$ and a G-equivariant map $f_{<k} : L_{<k} \to L$ which induces an isomorphism in homology when $r < k$ and has $H_r(L_{<k}) = 0$ when $r \geq k$, and what are the implications of this existence?

Equivariant Moore Approximation

Let G be an arbitrary group acting on a topological space X. Let k be any integer.

Definition

A G-equivariant Moore k-approximation of X is a G-space $X_{<k}$ and a G-equivariant map

$$f_{<k} : X_{<k} \to X$$

such that

- the induced map on homology

$$H_r(f_{<k}) : H_r(X_{<k}) \to H_r(X)$$

  is an isomorphism for all $r < k$, and
- $H_r(X_{<k}) = 0$, for all $r \geq k$.

Examples

Example

If the G-action on X has a fixed point $x \in X$. Then the inclusion $\{x\} \to X$ is a G-equivariant 1-approximation of X.

Example

Let G act on itself by left translation. Let $d \in \mathbb{Z}$ such that $H_d(G) \neq 0$. Then G does not admit an equivariant Moore k-approximation for any $k \leq d$.

Example

Every smooth 4-dimensional toric manifold admits a T^2-equivariant Moore k-approximation, for every k

Fiber Bundles

Consider a locally trivial fiber bundle $\pi : E \to B$ with fiber L and structure group G. The fiber L is a G-space. The bundle π induces a classifying map $b_\pi : B \to BG$ such that E is isomorphic to the pull-back of the Borel construction $EG \times_G L$ along the map b_π.

Geometric Fiberwise Truncation

Let $f_{<k} : L_{<k} \to L$ be a G-equivariant Moore k-approximation of L. Then we define the fiber bundle $\pi_{<k} : \mathrm{ft}_{<k}E \to B$ to be the pull-back of the Borel construction $EG \times_G L_{<k}$ along b_π. The map $f_{<k}$ induces a morphism of fiber bundles

$$F_{<k} : \mathrm{ft}_{<k}E \to E$$

which covers the identity map on B.

It should be noted that the bundle $\pi_{<k} : \mathrm{ft}_{<k}E \to B$ has fiber $L_{<k}$ and structure group G, and that $F_{<k}$ induces $f_{<k}$ when restricted to a fiber.

Serre Spectral Sequence

For such a fiber bundle $\pi : E \to B$ there is an associated cohomological spectral sequence, the Serre spectral sequence with

$$E_2^{p,q} = H^{p+q}\left(B; \mathcal{H}^q\left(L\right)\right)$$

which converges to $H^{p+q}\left(E\right)$. Here $\mathcal{H}^q\left(L\right)$ is the local coefficient system determined by the $\pi_1\left(B\right)$-action on L whose groups are the q-th cohomology groups of L.

The morphism $F_{<k}$ induces a morphism between the Serre spectral sequences $F_{<k,s}^{p,q} : E_s^{p,q} \to E_{<k,s}^{p,q}$ such that the following diagram commutes

$$\begin{array}{ccc}
E_s^{p,q} & \xrightarrow{F_{<k,s}^{p,q}} & E_{<k,s}^{p,q} \\
\downarrow d_s^{p,q} & & \downarrow d_{<k,s}^{p,q} \\
E_s^{p+s,q-s+1} & \xrightarrow[F_{<k,s}^{p+s,q-s+1}]{} & E_{<k,s}^{p+s,q-s+1}
\end{array}$$

Vanishing differentials

Lemma

Let $\pi : E \to B$ be a fiber bundle whose fiber is L and structure group is G. Assume that a G-equivariant Moore k-approximation of L exists. Then for every $s \geq 2$ and every $q > k$ such that $q - s + 1 \leq k$ we have that the differential

$$d_s^{p,q} : E_s^{p,q} \to E_s^{p+s,q-s+1}$$

vanishes.

Theorem

If the fiber L admits a G-equivariant Moore k-approximation for all $k \geq 1$ then the Serre spectral sequence collapses at the E_2-term, and we have an isomorphism for every $r \geq 0$

$$H^r(E) \cong \bigoplus_{q=0}^{r} H^q\left(B; \mathcal{H}^{r-q}(L)\right).$$

Proof the the lemma

We proceed inductively on $s \geq 2$. When know that when $s = 2$ we have for any p the commutative diagram where we assume $q = k + 1$.

$$\begin{array}{ccc}
H^{p+q}\left(B;\mathcal{H}^{q}\left(L\right)\right) & \xrightarrow{F_{<k,2}^{p,q}} & 0 \\
\downarrow{\scriptstyle d_2^{p,q}} & & \downarrow \\
H^{p+q+1}\left(B;\mathcal{H}^{q-1}\left(L\right)\right) & \xrightarrow{F_{<k,2}^{p+2,q-1}} & H^{p+q+1}\left(B;\mathcal{H}^{q-1}\left(L_{<k}\right)\right)
\end{array}$$

We know that the map $F_{<k,2}^{p+2,q-1}$ is an isomorphism. Therefore, $d_2^{p,q} = 0$. This implies that $F_{<k,3}^{p,q} : E_3^{s,t} \to E_{<k,3}^{p,q}$ is an isomorphism for all p and all $q \leq k$. But since this is true, the same logic as above implies that $d_3^{p,q} = 0$ whenever $q \geq k$ and $q - 2 \leq k$. Proceeding inductively this implies that $F_{<k,s}^{p,q}$ is an isomorphism for all $s \geq 2$ and all $q \leq k$. Therefore each such differential $d_s^{p,q} = 0$

Connections With Similar Work

Theorem ([Banagl, 2011])

Let L be a closed oriented Riemannian manifold, let $\pi : E \to B$ be a flat smooth fiber bundle over the closed, smooth base manifold B with fiber L and structure group the isometries of L. Then the Serre spectral sequence with real coefficients of $\pi : E \to B$ degenerates at the E_2-term.

The reason this is interesting is because the spectral sequence arguments used by Banagl are almost the same as those used above. The main difference being that he uses a differential forms approach. In doing so it is necessary to make the assumption that the bundle is flat and isometrically-structured in order to construct a fiberwise-cotruncated de Rham complex which then plays the role of our $\mathrm{ft}_{<k}E$.

Comparison

The following is an example of a group acting isometrically on a Riemannian manifold which does not admit an equivariant Moore k-approximation for every possible k.

Example

Let $\mathbb{Z}$ act on S^1 by rotating by an irrational angle $\Theta \in \mathbb{R} \setminus \mathbb{Q}$. This satisfies that S^1 is a Riemannian manifold and $\mathbb{Z}$ acts by isometries. However no $\mathbb{Z}$-equivariant Moore 1-approximation exists for such an action.

- This above action determines a flat fiber bundle $\pi : E_\Theta \to B\mathbb{Z} = S^1$, whose fiber is a S^1 with the above $\mathbb{Z}$-action.
- The real Serre spectral sequence of this bundle collapses at the E_2-term by Banagl's theorem.
- Our theorem cannot determine this because no equivariant Moore 1-approximation exists.

References

Banagl, Markus
Isometric group actions and the cohomology of flat fiber bundles.
Groups Geom. Dyn. 7 (2013), no. 2, 293321.