Remark 1.1. These notes are my interpretation of section 4.2.1 of Lück's 'A basic Introduction to Surgery Theory.' This section is titled: *Symmetric Forms and Surgery Kernels*. The goal is to introduce symmetric forms, and to show how such foms arise naturaly in surgery theory. In an effort to make the presentation as accesable as possible we arange the order of presentation slightly differently than Luëk and fill in some details of certain arguments.

2. Symmetric Forms

Definition 2.1. A ring with involution R is an associative ring R with unit 1 and a unital ring anti-homomorphism $\overline{-} : R \to R : r \mapsto \overline{r}$ such that $\left(\overline{-}\right)^2 = id_R$. In particular this means that $\overline{r \cdot s} = \overline{s} \cdot \overline{r}$, $\overline{\overline{r}} = r$, $\overline{r+s} = \overline{r} + \overline{s}$, and $\overline{1} = 1$.

Remark 2.2. Let M be a left R-module. We can give M the structure of a right R-module by defining the action of $s \in R$ on $r \in R$ by $r \cdot s = \overline{s} \cdot r$. The space $M^* = \hom_R(M, R)$ comes with a natural right R-module structure given by $(f \cdot r)(s) = f(s) \cdot r$ for $r, s \in R$ and $f \in M^*$. When R is a ring with involution, we can give M^* a canonical left R-module structure by $(r \cdot f)(s) = f(s) \cdot \overline{r}$. Thus, we can consider both M and M^* as left R-modules, or both as right R-modules.

The left R-module M comes equipped with the following canonical left R-module homomorphism

$$e(M) : M \to (M^*)^*$$

which is defined by evaluation

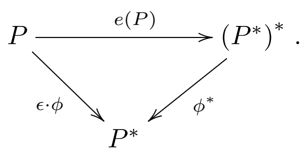
$$(e(M)(x))(f) = \overline{f(x)},$$

for every $f \in M^*$ and $x \in M$. Here we are viewing $(M^*)^*$ as a left module via the action

$$(r \cdot \alpha)(f) = \alpha(f) \cdot \overline{r}$$

for $r \in R$, $f \in M^*$ and $\alpha \in (M^*)^*$. This is the left R-modules structure defined using the involution on R and the left R-module structure on M^*.

Definition 2.3. Let $\epsilon \in Z(R)$ such that $\epsilon^2 = 1$. That is to say that ϵ is a central idempotent in R. An ϵ-symmetric form (P, ϕ) over R is a finitely-generated projective left R-module P together with a left R-module homomorphism $\phi : P \to P^*$, such that the following diagram commutes



The form (P, ϕ) is called non-degenerate if ϕ is an isomorphism.

Remark 2.4. Given an ϵ-symmetric form (P, ϕ) we can use the involution in R to give P the structure of both a left R-module and a right R-module. Using the left R-module structure on the left and the right on the right we can define the abelian group $P \otimes_R P$. Using this we can defined the following map

$$\lambda : P \otimes_R P \to R : (p, q) \mapsto \phi(p)(q).$$

Another way of writing this is that $\overline{\lambda(p, q)} = (e(P)(q))(\phi(p)) = \phi^*(e(P)(q))(p)$. Since the above diagram is required to commmute we have the follwing identity $\overline{\lambda(p, q)} = \epsilon \cdot \phi(q)(p) = \epsilon \cdot \lambda(q, p)$. So that

$$\lambda(q, p) = \epsilon \cdot \overline{\lambda(p, q)}.$$

This justifies the term ϵ-symmetric form. Notice that the R-linearity of ϕ implies, among other things, that $\lambda(r \cdot p, q) = \lambda(p, q) \cdot \overline{r}$, and $\lambda(p, r \cdot q) = r\lambda(p, q)$, and that λ is bi-linear.

If the form (P, ϕ) is non-degenerate then ϕ is an isomorphism, and hence the bilinear form λ is non-degenrate. That is to say that if $\lambda(p, q) = 0$ for all q, then $p = 0$.

Example 2.5. Let P be a finitely genearted projective left R-module. We define the standard hyperbolic ϵ-symmetric form $H^{\epsilon}(P) = (P \oplus P^*, \phi)$, where ϕ is defined as follows:

$$\phi(p, f)(q, g) = \epsilon \cdot f(q) + \overline{g(p)}$$

This is clearly additive in both the (p, f)-variable, and in the (q, g)-variable. We are viewing $P \oplus P^*$ as a left module with action defined by $r \cdot (p, f) = (r \cdot p, r \cdot f)$. Thus

$$\phi(r \cdot (p, f))(q, g) = \epsilon \cdot f(q) \cdot \overline{r} + \overline{g(p)} \cdot \overline{r} = (r \cdot \phi(p, f))(q, g),$$

and

$$\phi(p, f)(r \cdot (q, g)) = r \cdot \phi(p, q)(q, g)$$

So that ϕ is a left R-module homomorphism.

We now verify that the desired diagram commutes. We have that

$$\phi^*(e(P \oplus P^*)(p, f))(q, g) = e(P \oplus P^*)(p, f)(\phi(q, g)) = \overline{\phi(q, g)(p, f)}$$

Using the definition of ϕ and the fact that $\epsilon \cdot f(q) = f(q)\overline{\epsilon}$ we have that this becomes

$$\overline{\phi(q, g)(p, f)} = \overline{\epsilon \cdot g(p) + \overline{f(q)}} = \overline{g(p)}\overline{\epsilon} + f(q) = \left(f(q)\overline{\epsilon} + \overline{g(p)}\right)\overline{\epsilon} = \epsilon \cdot \phi(p, f)(q, g).$$

This shows that the desired digram commutes. Therefore the $H^{\epsilon}(P)$ is an ϵ-symmetric form.

After identifying $(P \oplus P^*)^* \cong P^* \oplus (P^*)^*$ we can write ϕ as a composition of the matrix

$$\begin{pmatrix} 0 & 1 \\ \epsilon & 0 \end{pmatrix} : P \oplus P^* \to P^* \oplus P$$

with the map

$$1 \oplus e(P) : P^* \oplus P \to P^* \oplus (P^*)^*$$

Thus ϕ is an isomorphism if and only if $e(P)$ is an isomorphism. This is true when P is finitely-generated and projective. Thus the standard hyperbolic ϵ-symmetrc form $H^{\epsilon}(P)$ is non-degenerate.

3. Surgery Kernels

Remark 3.1. Let M be a connected closed manfiold of dimension n, and X be a connected finite Poincaé complex of dimension n. Fix a vector bundle $\xi : E \to X$. A normal map for ξ is a pair of morphism $\left(\overline{f}, f\right)$ sucht that $f : M \to X$ is a continuous map, and $\overline{f} : TM \oplus \underline{\mathbb{R}}^a \to E$ is a bundle morphism covering f, for some $a \geq 0$. Here $\underline{\mathbb{R}}^a$ denotes the trivial a-dimensional vector bundle. We assume that this normal map has degree 1. That is to say that $f_*[M] = [X]$, where $[M]$ is the fundamental class of M and $[X]$ is the fundamental class of X. Choose a base point $b \in M$, we further assume that $f_* : \pi_1(M, b) \to \pi_1(X, f(b))$ is an isomorphism. Let$\tilde{b} \in \tilde{M}$ and $\widetilde{f(b)} \in \tilde{X}$ be choices of lifts of b and $f(b)$ to the respective universal cover. This determines a lift $\tilde{f} : \tilde{M} \to \tilde{X}$ which is equivariant with respect to the action of the respective fundamental groups. We will now define the surgery kernel associated to the above data, and we will show that it gives rise to an ϵ-symmetric form over the group ring $\mathbb{Z}\pi$. Here we identify $\pi = \pi_1(M, b) \cong \pi_1(X, f(b))$, and consider $\tilde{f}$ as a π-equivariant map.

The map $\tilde{f}$ induces two long exact sequences of $\mathbb{Z}\pi$-modules

$$\longrightarrow H^{n-r}\left(\tilde{f}\right) \xrightarrow{j^*} H^{n-r}\left(\tilde{X}\right) \xrightarrow{\tilde{f}^*} H^{n-r}\left(\tilde{M}\right) \xrightarrow{\delta^*} H^{n-r+1}\left(\tilde{f}\right) \longrightarrow$$

and

$$\longrightarrow H_{r+1}\left(\tilde{f}\right) \xrightarrow{\delta_*} H_r\left(\tilde{M}\right) \xrightarrow{\tilde{f}_*} H_r\left(\tilde{X}\right) \xrightarrow{j_*} H_r\left(\tilde{f}\right) \longrightarrow$$

The normal map $\left(\overline{f}, f\right)$ has degree 1, and therefore the cap product with the fundamental class induces the following commutative diagram of $\mathbb{Z}\pi$-modules, in which the vertical maps are isomorphisms

$$\begin{array}{ccc} H^{n-r}\left(\tilde{M}\right) & \xleftarrow{\tilde{f}^*} & H^{n-r}\left(\tilde{X}\right) \\ \downarrow{\scriptstyle -\cap[M]} & & \downarrow{\scriptstyle -\cap[X]} \\ H_r\left(\tilde{M}\right) & \xrightarrow{\tilde{f}_*} & H_r\left(\tilde{X}\right) \end{array}$$

This implies that $\tilde{f}^*$ is injective and $\tilde{f}_*$ is surjective. It follows that the maps j^* and j_* both vanish, and we have the following short exact sequences for every $r \geq 0$

$$0 \longrightarrow H^{n-r}\left(\tilde{X}\right) \xrightarrow{\tilde{f}^*} H^{n-r}\left(\tilde{M}\right) \xrightarrow{\delta^*} H^{n-r+1}\left(\tilde{f}\right) \longrightarrow 0$$

and

$$0 \longrightarrow H_{r+1}\left(\tilde{f}\right) \xrightarrow{\delta_*} H_r\left(\tilde{M}\right) \xrightarrow{\tilde{f}_*} H_r\left(\tilde{X}\right) \longrightarrow 0$$

Definition 3.2. The homological surgery kernel of the normal map $\left(\overline{f}, f\right)$ is defined for each $r \geq 0$ to be

$$K_r\left(\tilde{M}\right) = H_{r+1}\left(\tilde{f}\right)$$

and the cohomological surgery kernel of the normal map $\left(\overline{f}, f\right)$ is defined for each $r \geq 0$ to be

$$K^{n-r}\left(\tilde{M}\right) = H^{n-r+1}\left(\tilde{f}\right)$$

Furthermore, since $-\cap[M]$ is an isomorphism, we have the $\mathbb{Z}\pi$-module morphism $\Phi_* : H_{r+1}\left(\tilde{f}\right) \to H^{n-r+1}\left(\tilde{f}\right)$ defined to be

$$\Phi_* = \delta^* \circ \left(- \cap [M]\right)^{-1} \circ \delta_*$$

which makes the following diagram commute

$$\begin{array}{ccccccccc} 0 & \longleftarrow & K^{n-r}\left(\tilde{M}\right) & \xleftarrow[\delta^*]{} & H^{n-r}\left(\tilde{M}\right) & \xleftarrow{\tilde{f}^*} & H^{n-r}\left(\tilde{X}\right) & \longleftarrow & 0 \\ & & {\scriptstyle \Phi_*}\uparrow & & \downarrow{\scriptstyle -\cap[M]} & & \downarrow{\scriptstyle -\cap[X]} & & \\ 0 & \longrightarrow & K_r\left(\tilde{M}\right) & \xrightarrow{\delta_*} & H_r\left(\tilde{M}\right) & \xrightarrow{\tilde{f}_*} & H_r\left(\tilde{X}\right) & \longrightarrow & 0 \end{array}$$

A simple diagram chasing agrument shows that Φ_* is an isomorphism for all $r \geq 0$.

Proposition 3.3. *(1) The cap product $- \cap [M]$ induces an isomorphism*

$$- \cap [M] : K^{n-r}\left(\tilde{M}\right) \to K_r\left(\tilde{M}\right)$$

for all $r \geq 0$.

(2) If f is r-connected then there is are natural $\mathbb{Z}\pi$-module isomorphism

$$\pi_{r+1}(f) \cong \pi_{r+1}\left(\tilde{f}\right) \cong H_{r+1}\left(\tilde{f}\right) = K_r\left(\tilde{M}\right).$$

The first induced by the universal cover, and the second is the Hurewicz homomorphism.

Proof. Item 1 is proven in the paragraph before the proposition. The second isomorphism in item 2 follows from the relative Hurewicz theorem since f is r-connected. The fist isomorphism in item 2 follows from the long exact sequence of homotopy groups of a fibration. We apply this long exact sequence to the universal covers of M and X. Since the fibers are discrete we have that the homotopy groups of the universal cover are isomorphic to those of the base whenever we are in degree > 1. The result now follows from the five-lemma. □

Remark 3.4. We now assume that $\tilde{f}$ is k-connected and that $n = 2k$, with $k \geq 2$. In this case we will show that the surgery kernel defined above gives rise to a $(-1)^k$-symmertric form over $\mathbb{Z}\pi$ with the w-twisted involution. We need first to define choose the appropriate $\mathbb{Z}\pi$-module and define a duality morphism.

Recall that there is a Kronecker product

$$\langle , \rangle : H^k\left(\tilde{M}\right) \times H_k\left(\tilde{M}\right) \to \mathbb{Z}\pi$$

which is induced by the evaluation pairing

$$\hom_{\mathbb{Z}\pi}\left(C_p\left(\tilde{M}, \mathbb{Z}\pi\right)\right) \times C_p\left(\tilde{M}\right) \to \mathbb{Z}\pi : (\alpha, x) \mapsto \alpha(x).$$

This paring induces a pairing

$$\langle , \rangle : K^k\left(\tilde{M}\right) \times K_k\left(\tilde{M}\right) \to \mathbb{Z}\pi.$$

This is well defined because of the identity

$$\left\langle \tilde{f}^*(\alpha), \delta_*(x)\right\rangle = \tilde{f}^*(\alpha)(\delta_*(x)) = \delta^*\tilde{f}^*(\alpha)(x) = 0.$$

Since $n = 2k$ we have the isomorphism

$$\Phi_* : K_k\left(\tilde{M}\right) \to K^k\left(\tilde{M}\right)$$

Thus we have a well defined pairing

$$s = \langle , \rangle \circ (\Phi_* \times 1) : K_k\left(\tilde{M}\right) \times K_k\left(\tilde{M}\right) \to \mathbb{Z}\pi.$$

Which, for $x, y \in K_k\left(\tilde{M}\right)$, takes the value

$$s(x, y) = \Phi_*(y)(x)$$

This gives rise to

$$\phi : K_k\left(\tilde{M}\right) \to K_k\left(\tilde{M}\right)^* : x \mapsto \Phi_*(x),$$

which is a $\mathbb{Z}\pi$-module homomorphism since it is the restriction of the inverse of $- \cap [M]$ which is constructed to be a $\mathbb{Z}\pi$-module homomorphism.

Now we have a candidate $\left(K_k\left(\tilde{M}\right), \phi\right)$ for an $(-1)^k$-symmetric form over $\mathbb{Z}\pi$. We still need to verify that ϕ satisfies the relation $\Phi_*(x)(y) = (-1)^k \Phi_*(y)(x)$, and that $K_k\left(\tilde{M}\right)$ is a finitely generated projective $\mathbb{Z}\pi$-module

Definition 3.5. Let $I_k\left(\tilde{M}\right)$ be the set of pointed homotopy classes of pointed immersions from S^k to $\tilde{M}$.

To see that this is $(-1)^k$-symmetric, we identify the groups

$$K_k\left(\tilde{M}\right) \cong I_k\left(\tilde{M}\right),$$

and the pairing s with the pairing

$$\lambda : I_k\left(\tilde{M}\right) \times I_k\left(\tilde{M}\right) \to \mathbb{Z}\pi,$$

which we know to be $(-1)^k$-symmetric. The following proposition does this for us.

Proposition 3.6. *There is a $\mathbb{Z}\pi$-module homomorphism*

$$t_k : K_k\left(\tilde{M}\right) \to I_k\left(M\right)$$

which makes the following diagram commute

$$\begin{array}{ccc} K_k\left(\tilde{M}\right) \times K_k\left(\tilde{M}\right) & & \\ \Big\downarrow{\scriptstyle t_k \times t_k} & \overset{s}{\searrow} & \mathbb{Z}\pi \\ & \underset{\lambda}{\nearrow} & \\ I_k\left(\tilde{M}\right) \times I_k\left(\tilde{M}\right) & & \end{array}$$

Proof. Let $x \in K_k\left(\tilde{M}\right)$, using the inverse of the Hurewicz isomorphism we identify x as an element of $\pi_{k+1}\left(\tilde{f}\right)$. By theorem 3.59 parts 1 and 2 we may represent x by a commutative diagram

$$\begin{array}{ccc} S^n & \xrightarrow{q} & M \\ \Big\downarrow & & \Big\downarrow{\scriptstyle f} \\ D^{n+1} & \xrightarrow{Q} & X \end{array}$$

where, among other things, q is an immersion whose regular homotopy class depends only on the class x. The map t_k is defined by sending x to the class $[q]$.

The Hurewics homomorphism fits into the following commutative diagram

$$\begin{array}{ccc} \pi_{k+1}\left(\tilde{f}\right) & \xrightarrow{\delta_{\#}} & \pi_k\left(\tilde{M}\right) \\ \Big\downarrow{\scriptstyle h_{\tilde{f}}} & & \Big\downarrow{\scriptstyle h_{\tilde{M}}} \\ H_{k+1}\left(\tilde{f}\right) & \xrightarrow{\delta_*} & H_k\left(\tilde{M}\right) \end{array}.$$

If we view the class $[q]$ as an element of $\pi_k\left(\tilde{M}\right)$ then we have that $\delta_{\#}(x) = [q]$. We already know that $h_{\tilde{f}}$ is an isomorphsm and δ_* is injective. Thus the element

$$\delta \circ h^{\tilde{f}}(x) = h_{\tilde{M}} \circ \delta_{\#}(x) = h_{\tilde{M}}[q]$$

is non-zero whenever x is non-zero. That is to say for $0 \neq x \in K_k\left(\tilde{M}\right)$ the homology class $\delta_*(x)$ can be represented by an immersion $q : S^k \to \tilde{M}$. Given $x, y \in K_k\left(\tilde{M}\right)$ the pairing $\Phi_*(x)(y)$ is given by evaluating the Poincaré dual of x on the element y. This gives the same value as taking the cup product of the Poincaré dual of $\delta_*(x)$ with the Poincaré dual of $\delta_*(y)$ and evaluating the resulting product on $[M]$. It can be shown that this is exactly the intersection number of the immersions representing $\delta_*(x)$ and the immersion representing $\delta_*(y)$. But this is exactly the pairing λ applied to $t_k(x)$ and $t_k(y)$. Thus the diagram commutes. □

Remark 3.7. We will now show that $K_k\left(\tilde{M}\right)$ is a finitely generated projective $\mathbb{Z}\pi$-module. We will prove a stronger statement, that $K_k\left(\tilde{M}\right)$ is finitely generated and stably free. This has the consequence that in the reduced Grothendeick group of finitely generated projective $\mathbb{Z}\pi$-module, $\tilde{K}_0(\mathbb{Z}\pi)$ the class $[K_k\left(\tilde{M}\right)] = [0]$.

Definition 3.8. A finitely generated R-module V is stably free if there are non-negative integers l and m and an isomorphism

$$V \oplus R^l \cong R^m.$$

Proposition 3.9. *If $f : X \to Y$ is k-connected for $n = 2k$, or $2k+1$, then $K_k\left(\tilde{M}\right)$ is finitely generated and stably free.*

Proof. Since M is a smooth compact manifold we know that $H^k\left(\tilde{M}\right)$ is finitely generated over $\mathbb{Z}\pi$, thus there is an integer $l \geq 0$ and a surjective map

$$(\mathbb{Z}\pi)^l \to H^k\left(\tilde{M}\right) \to K^k\left(\tilde{M}\right) \cong K_k\left(\tilde{M}\right),$$

where the middle map comes from the short exact sequence above. Therefore $K_k\left(\tilde{M}\right)$ is finitely generated. We need now to see that it is projective.

Consider the chain complex $C_*\left(\tilde{f}\right)$, This is the cellular chain complex of the mapping cone on f. Thus each $C_k\left(\tilde{f}\right)$ is a finitely generated free $\mathbb{Z}\pi$-module. Since $\tilde{f}$ is k-connected we have that $\operatorname{im} d_r = \ker d_{r-1}$ for all $r \leq k$. For each $r \leq k+1$ we have the following exact sequence

$$\ker d_r \longrightarrow C_r\left(\tilde{f}\right) \longrightarrow \ker d_{r-1} \longrightarrow 0$$

Note that $C_0\left(\tilde{f}\right) = \ker d_0$ is a finitely generated free $\mathbb{Z}\pi$-module. The above exact sequence allows us to proceed inductively to deduce that $\ker d_r$ is a finitely generated $\mathbb{Z}\pi$-module.

The inclusion $i : \ker d_{k+1} \to C_{k+1}\left(\tilde{f}\right)$ induces a chain homotopy equivalence $i_* : D_* \to C_*\left(\tilde{f}\right)$, where D_* is the chain complex

$$\cdots \longrightarrow C_{k+2}\left(\tilde{f}\right) \xrightarrow{d_{k+2}} \ker d_{k+1} \longrightarrow 0 \longrightarrow \cdots$$

Similarly we can consider the chain complex $C^{n+1-*}\left(\tilde{f}\right)$. The fact that $\tilde{f}$ is k-connected again impliex that we can $C^{n+1-*}\left(\tilde{f}\right)$ is chain homotopy equivalent to a chain complex E^{n+1-*} given by

$$\cdots \longrightarrow C^{n+1-(k+1)}\left(\tilde{f}\right) \xrightarrow{d^{n+1-(k+1)}} \ker d^{n+1-k} \longrightarrow 0 \qquad \cdots$$

where $\ker d^{n+1-k}$ is in the degree $k+1$ position. The duality morphism Φ_* defined above is induced by a chain morphism, and thus induced a chain homotopy equivalence $C_{*+1}\left(\tilde{f}\right) \simeq C^{n+1-*}\left(\tilde{f}\right)$. Thus we have the sequence of chain homotopy equivalences

$$\hom_{\mathbb{Z}\pi}\left(C_{*+1}\left(\tilde{f}\right), \operatorname{im} d_{k+2}\right) \simeq \hom_{\mathbb{Z}\pi}\left(C^{n+1-*}\left(\tilde{f}\right), \operatorname{im} d_{k+2}\right) \simeq \hom_{\mathbb{Z}\pi}\left(E^{n+1-*}, \operatorname{im} d_{k+2}\right)$$

Since $E^{n+1-r} = 0$ for all $r \leq k$, we have that the cohomology group

$$H^{k+2}\left(\hom_{\mathbb{Z}\pi}\left(C_{*+1}\left(\tilde{f}\right), \operatorname{im} d_{k+2}\right)\right) = 0$$

This implies that the following sequence is exact at the middle term

$$\hom_{\mathbb{Z}\pi}\left(C_{k+1}\left(\tilde{f}\right), \operatorname{im} d_{k+2}\right) \xrightarrow{d^{\#}_{k+2}} \hom_{\mathbb{Z}\pi}\left(C_{k+2}\left(\tilde{f}\right), \operatorname{im} d_{k+2}\right) \xrightarrow{d^{\#}_{k+1}} \hom_{\mathbb{Z}\pi}\left(C_{k}\left(\tilde{f}\right), \operatorname{im} d_{k+2}\right)$$

The map d_{k+2} is an element in the middle term and $d^{\#}_{k+1}(d_{k+2}) = 0$ thus there exists a map $\phi : C_{k+1}\left(\tilde{f}\right) \to \operatorname{im} d_{k+2}$ such that $\phi \circ d_{k+2} = d_{k+2}$.

Let $j : \operatorname{im} d_{k+2} \to C_{k+1}\left(\tilde{f}\right)$ be the inclusion. We know that $j \circ d_{k+2} = d_{k+2}$. Thus we have

$$\phi \circ j \circ d_{k+2} = \phi \circ d_{k+2} = d_{k+2}.$$

Since d_{k+2} is surjective onto its image we have that

$$\phi \circ j = id_{\operatorname{im} d_{k+2}}$$

Therefore $\operatorname{im} d_{k+2}$ is isomorphic to a direct summand of $C_{k+1}\left(\tilde{f}\right)$, and hence it is projective. Thus $\operatorname{im} d_{k+2}$ is also isomorphic to a direct summand of $\ker d_{k+1}$, and its complimentary summand is isomorphic to $H_{k+1}\left(\tilde{f}\right) = K_k\left(\tilde{M}\right)$. Hence we have that $K_k\left(\tilde{M}\right)$ is a finitely generated projective $\mathbb{Z}\pi$-module.

We note that this means that

$$C_{k+1}\left(\tilde{f}\right) \cong \ker d_{k+1} \oplus \operatorname{im} d_{k+1} \cong \operatorname{im} d_{k+2} \oplus H_{k+1}\left(\tilde{f}\right) \oplus \operatorname{im} d_{k+1}.$$

Furthermore by Poincare duality and the fact that $\tilde{f}$ is k-connected we have that the chain complexes

$$\cdots \longrightarrow C_{k+2}\left(\tilde{f}\right) \longrightarrow \operatorname{im} d_{k+2} \longrightarrow 0$$

and

$$0 \longrightarrow \operatorname{im} d_{k+1} \longrightarrow C_{k-1}\left(\tilde{f}\right) \longrightarrow \cdots$$

are both acyclic, and hence so is their direct sum A_*. For an acyclic complex the sum of the odd terms is isomorphic to the sum of the even terms. In this case we have

$$\operatorname{im} d_{k+2} \oplus \operatorname{im} d_{k+1} \oplus \left(\bigoplus_{i \neq 0} A_{k+1+2i}\right) \cong \bigoplus A_{k+2i}.$$

Now adding $H_{k+1}\left(\tilde{f}\right)$ to both sides gives on the left side

$$H_{k+1}\left(\tilde{f}\right) \oplus \operatorname{im} d_{k+2} \oplus \operatorname{im} d_{k+1} \oplus \left(\bigoplus_{i \neq 0} A_{k+1+2i}\right) \cong C_{k+1}\left(\tilde{f}\right) \cong \bigoplus C_{k+1+2i}\left(\tilde{f}\right).$$

and on the right

$$H_{k+1}\left(\tilde{f}\right) \oplus \left(\bigoplus A_{k+2i}\right) \cong H_{k+1}\left(\tilde{f}\right) \oplus \left(\bigoplus C_{k+2i}\left(\tilde{f}\right)\right)$$

Each $C_r\left(\tilde{f}\right)$ is a free $\mathbb{Z}\pi$-module, thus $K_k\left(\tilde{M}\right) = H_{k+1}\left(\tilde{f}\right)$ is stably free. $\square$