BORDISM CATEGORIES AND TFT'S

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Abstract.

Contents

1. Motivation

Remark 1.1. We begin by examining some classical invariants associated to topological spaces. For convenience we will restrict our attention to smooth manifolds. As such, in this paper we consider only smooth manfiolds with smooth maps between them.

In this first section we examine certain topological invariants of compact smooth manfiolds without boundary that have the following two properties:

(1) These invariants vanish on any manifold that is the boundaries of another manifold, and
(2) These invariants are additive over direct sums.

These two properties ensure that such topological invariants are in fact bordism invariants. These will serve as motivation as to why we would want to define and study the bordism category.

1.1. **The Euler Characteristic.**

Definition 1.2. Let $\mathbb{K}$ be a field, and $\eta : \mathbb{Z} \to \mathbb{K}$ the map sending $1_{\mathbb{Z}}$ to 1_K. For a topological space X and a positive integer k, the k-th Betti number of X is the number

$$\beta_k(X; \mathbb{K}) = \dim_{\mathbb{K}} H_k(X; \mathbb{K}).$$

The Euler characteristic of X is the number

$$\chi_{\mathbb{K}}(X) = \eta\left(\sum_{i\geq 0} (-1)^i \beta_i(X; \mathbb{K})\right) \in \mathbb{K}.$$

Remark 1.3. Let M be a connected smooth manfiold of dimension n, and $\mathbb{K} = \mathbb{Z}/2$. We know that M has $H_n(M; \mathbb{Z}/2) \cong \mathbb{Z}/2$ is generated by the fundamental class $[M]$, and $H_r(M; \mathbb{Z}/2) = 0$ for $r > n$, and $r < 0$.

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euler1 **Proposition 1.4.** *Assum that $M = \partial N$ is the boundary of a smooth compact $n+1$-dimensional manifold N. Then*

$$\chi_{\mathbb{Z}/2}(M) = 0 \in \mathbb{Z}/2.$$

Proof. Assume that $M = \partial N$ is the boundary of some $n+1$ dimensional manifold. Let $DN = N \cup_M N$ be the double of N obtained by gluing two coppies of N together along their common boundary M. The Mayer-Vietoris property of homology now shows that the Euler characteristic of DN is

$$\chi_{\mathbb{Z}/2}(DN) = 2\chi_{\mathbb{Z}/2}(N) - \chi_{\mathbb{Z}/2}(M) = \chi_{\mathbb{Z}/2}(M) \in \mathbb{Z}/2.$$

We also know that the Euler characteristic of a compact odd dimensional manifold without boundary is zero. Thus when n is odd then $\chi_{\mathbb{Z}/2}(M) = 0$. If n is even then $n+1$ is odd and $\chi_{\mathbb{Z}/2}(DN) = 0$. Thus we have that $\chi_{\mathbb{Z}/2}(M) = 0 \in \mathbb{Z}/2$. □

euler2 **Corollary 1.5.** *If $M = M_1 \sqcup M_2 = \partial N$ the boundary of some manifold N, then*

$$\chi_{\mathbb{Z}/2}(M_1) = \chi_{\mathbb{Z}/2}(M_2).$$

Proof. This follows because the Euler characteristic is additive over disjoint unions $\chi_{\mathbb{K}}(M_1 \sqcup M_2) = \chi_{\mathbb{K}}(M_1) + \chi_{\mathbb{K}}(M_2)$. □

1.2. **The Signature.**

Remark 1.6. For any compact smooth n-dimensional manifold M without boundary, an orientation of M is a homology class $[M] \in H_n(M; R)$ such that for every point $x \in M$ the restriction $H_n(M) \to H_n(M, M \setminus x) \cong H_{n-1}(S^{n-1}) \cong \mathbb{Z}$ maps $[M]$ to the generator of $H_n(M, M \setminus x)$. If such an orientation exists, the we say that M is oriented.

If M is oriented then M satisfies poincaré duality over $\mathbb{Q}$. That is to say: There is an isomorphism

$$D_M^R : H^r(M; \mathbb{Q}) \cong H_{n-r}(M; \mathbb{Q}).$$

If $n = 2m$ is even, then in degree m we have, by composing with the universal coefficients theorem, the isomorphism

$$D : H^m(M; \mathbb{Q}) \cong H_m(M; \mathbb{Q}) \cong H^m(M : \mathbb{Q})$$

This in turn defines a nondegenerate bilinear form

$$B_M^R : H^m(M; Q) \otimes H^m(M; Q) \to Q$$

which is symmertric if m is even.

To each nondegenerate symmetric bilinear form $B : V \otimes V \to \mathbb{Q}$ defined on a $\mathbb{Q}$-vector space V, one can define the signature $\sigma(B)$ of B to be the number of positive eigenvalues of $\hat{B}$ minus the number of negative eigenvalues of $\hat{B}$, where $\hat{B} : V \to V^\dagger$ is the adjoint of B, and $V^\dagger$ is the linear dual of V. This signature is an integer.

Thus, in our case we have the signature $\sigma(B_M) \in \mathbb{Z}$ of B_M, whenever $n = 4k$ is a multiple of 4. If $n \neq 0 \pmod 4$, then we define the signature to be 0. It is easy to see that $\sigma(B_M^R)$ is additive over disjoint union since homology is. We now examine the consequences of assuming that $M = \partial N$ is the boundary of some compact smooth manifold N of dimension $4n+1$.

Proposition 1.7. *Let M be a compact smooth oriented $4m$-dimensional manfiold without boundary. If $M = \partial N$ for some compact smooth oriented $4n+1$ dimensional manifold N then the signature of M vanishes*

$$\sigma(B_M) = 0.$$

Proof. We consider the long exact sequence of cohomology groups associated to the piar $(N, \partial N = M)$.

$$\cdots \xrightarrow{j^{2k}} H^{2k}(N;\mathbb{Q}) \xrightarrow{i^{2k}} H^{2k}(M;\mathbb{Q}) \xrightarrow{\delta^{2k}} H^{2k}(N,M;\mathbb{Q}) \xrightarrow{j^{2k+1}} \cdots.$$

Since these are $\mathbb{Q}$-vector spaces, we may choose a splitting of δ^{2k}, and by exactness we get an isomorphism

$$H^{2k}(M;\mathbb{Q}) \cong \operatorname{coker} j^{2k} \oplus \ker j^{2k+1}.$$

By degree consideration, under the pairing B_M elements of $\operatorname{coker} j^{2k}$ must pair with elements of $\ker j^{2k+1}$, and vice-versa. In particular, since B_M is nondegenerate, we have $\operatorname{coker} j^{2k} = \left(\operatorname{coker} j^{2k}\right)^{\perp}$, and $\ker j^{2k+1} = \left(\ker j^{2k+1}\right)^{\perp}$. By symmetry, and the univeral coefficients theorem, we have that the dual of $\operatorname{coker} j^{2k}$ is $\ker j^{2k+1}$, and vice-versa. Thus $\dim_{\mathbb{Q}} \ker j^{2k+1} = \dim_{\mathbb{Q}} \operatorname{coker} j^{2k}$. Thus $\ker j^{2k+1}$ forms a Lagrangian subspace of $H^{2k}(M;\mathbb{Q})$ with respect to B_M. This means that $\sigma(B_M) = 0$. ☐

Corollary 1.8. *If a compact smooth oriented manifold N has boundary $\partial N = M_1 \sqcup M_2$ then*

$$\sigma(D_{M_1}) = \sigma(D_{M_2}).$$

Remark 1.9. In the above statement we note that the orientation of ∂N is determined uniquely by the orientation of N.

This shows that the signature of compact oriented smooth manfiold without boundary is a bordism invariant. It should be pointed out though that this bordism is slightly different from the bordism we used for the Euler characteristic. That is to say, in our discussion on the Euler characteristic we never need the notion of orientation, and we did not require any of our manifolds to be oriented. However, for the signature, it is necessary to have an orientation on an manifold, and furthermore it is necessary for the bordism between manifolds to also be oriented. Thus we see the need to different types of bordisms depending on what it is we want to study.

2. Bordism

In this section we make it precise what is meant by bordism. We have already seen naievly that a bordism is jsut a manifold with boundary. What needs to be stressed is that a bordism should be thought of as a morphism between its boundary components. This means that we need to specify the boundary components as either inputs or outputs. So that we may specify the doman and codomain of the morphism. Something else that must be considered is how one can define composition of morphism. This amounts to gluing bordisms along common boundaries. Thus a bordism is a compact smooth manifold with boundary, along with a labeling of the boundary components as either inputs or outputs, and data specifying how to glue bordism together. This data useually come in the form of collar data. That is, we specify an open collar neighborhood of the boundary along which we can glue.

`bordism` **Definition 2.1.** Let M_1, M_2 be compact smooth manifolds without boundary of dimension n. A bordism from M_1 to M_2 is a triple $(N, \phi, \theta_1, \theta_2)$, where N is a smooth compact $n+1$-dimensional manifold with boundary ∂N, $\phi : \partial M \to M_1 \sqcup M_2$ a diffeomorphism, and $\theta_1 : [0,1) \times M_1 \to N$, $\theta_2 : (-1, 0] \to N$ are smooth embeddings such that $\phi \circ \theta_i|_{0 \times M_i} : M_i \to M_i$ is a the identity, for $i = 1, 2$.

Remark 2.2. If we are given a manifold M with boundary ∂M, then we define a bordism by specifying a map $c : \pi_0(\partial M) \to \{1, 2\}$. Then $c^{-1}(1)$ will be the domain of the bordism, and $c^{-1}(2)$ will be the codomain. One can show that for every manifold with nonempty boundary, the boundary has a collar neighborhood. Thus we can specify the maps θ_i, for $i = 1, 2$ once we have the map c to tell us whether to use the interval $(-1, 0]$ or $[0, 1)$.

Therefore a single manifold with boundary can be viewed as several bordisms. For example $[0, 1] \times M$ can be viewed as a morphism from M to M, from $\emptyset$ to $M \sqcup M$, or from $M \sqcup M$ to $\emptyset$ depending on how we choose c.

Proposition 2.3. *For smooth compact closed n-manifolds, we say that the diffeomorphism class $[M]$ of M is equivalent to the diffeomorphism class $[N]$ of N, $[M] \sim [N]$, if there is a bordism from M to N. This defines an equivlanece relation on the set of diffeomorphism classes of smooth manifolds.*

Proof. This is clearly symmetric and reflexive. To see that it is transitive note that one can glue the out going collar of a bordism to the incoming collar of another bordism, provided they are diffeomorphic. □

Definition 2.4. Let $\mathfrak{N}_n$ denote the set of equivalence classes of diffeomorphism classes of smooth manifolds under the relation of bordism. Define $[M] + [N] = [M \sqcup N]$. This defines a symmetric associative operation on $\mathfrak{N}_n$ with zero element $0 = [\emptyset]$

Remark 2.5. Notice that $M \sqcup M$ is bordant to $\emptyset$, so that $[M] + [M] = 0$. Thus every element in $\mathfrak{N}_n$ has order 2, and $-[M] = [M]$ is the inverse to $[M]$. Thus $\mathfrak{N}_n$ is a group. We call this the n-bordism group.

We can see already from euler1 1.4, and euler2 1.5 that the Euler characteristic is bordism invariant. Thus for any equivalence class $[M] \in \mathfrak{N}_n$, the assignment $[M] \mapsto \chi_{\mathbb{Z}/2}(M)$ is well defined. Furthermore this assignment is additive. Hence we have a well defined group homomorphism

$$\chi_{\mathbb{Z}/2} : \mathfrak{N}_n \to \mathbb{Z}/2 : [M] \mapsto \chi_{\mathbb{Z}/2}(M).$$

Such a map exists for every $n \geq 0$. Such a homomorphism is sometimes called a genus.

Definition 2.6. The n-bordism category Bord_n is the category whose objects are smooth closed compact n-dimensional manifolds. A morphism between two such manfiolds, from M_1 to M_2, is a diffeomorphism class of bordism, bordism 2.1, from M_1 to M_2, where the diffeomorphisms between bordisms are required to commute with the end maps θ_1 and θ_2.

Remark 2.7. For the manfiold M the identity map is give by the cobordism $M \times [0, 1]$, with end maps θ_1 and θ_2 the inclusion, and the inclusion shifted by $+1$, respectively. The map ϕ is the inclusion of M at each end.

Composition of morphism is given in the same way that the transitivity of $\sim$ was proven. For a bordism from M_1 to M_2 and a bordism from M_2 to M_3 we can glue the outgoing end of M_1 to the incoming end of M_2. This can be done in a smooth way so that the smooth structure on the resulting manfiold is unique up to diffeomorphism. This is the reason why we require morphisms to be diffeomorphism classes of bordisms, rather than just bordisms themselves.

This category can be give the structure of a symmetric monoidal category. The unit object is the empty n-dimensional manfiold, and the product structure is given by disjoint union.

3. Bordisms of Tangential Structures

Definition 3.1. A classifying space for a topological groups G is a fiber bundle $EG \to BG$ such that there is a $1-1$ correspondence between homotopy classes of maps $M \to BG$ and isomorhism classes of principle G-bundles $E \to M$, for any compact manifold M.

Remark 3.2. We are mostly concerned with the Lie groups O_n and O. The group O_n is the group of $n \times n$-matricies with real entries that are orthoginal with respect to the usual inner product on $\mathbb{R}^n$. One can include O_n into O_{n+1} by taking a matrix A to the block matric $I_1 \oplus A$, where I_1 is the 1×1 identity matrix. The colimit is defined to be the infinite orthoginal group O.

Definition 3.3. An n-dimensional tangential structure is a topological space Θ_n and a fibration $\pi_n : \Theta_n \to BO_n$. A stable tangential structure is a topological space Θ and a fibration $\pi : \Theta \to BO$. Since O is the colimit of the O_n we can always pullback a stable tanential structure to an n-dimensional tangential structure.

If M is an r-dimensional manifold, let $\hat{T}M = \underline{\mathbb{R}}^{n-r} \oplus TM$ be the Whitney sum of the tangent bundle and the trivial $n-r$ bundle on M. Let $\hat{b}_{TM} : M \to BO_n$ the classifying map for $\hat{T}M$. A Θ_n structure on M is a lift of $\hat{b}_{TM}$ over π_m to a map $\tilde{b}_{TM} : M \to \Theta_m$. Similarly, a Θ sutrcture on M a family of Θ_n structures for each n that are coherent with respect to the colimit maps.

Example 3.4. An orientation is the stable tangential structure $\Theta = BSO$, where SO is the infinite special orthoginal group, in this case the pullback Θ_n of Θ to BO_n is the classifying space BSO_n of the special orthoginal group of real $n \times n$-matricies.

Notice that a lift of $\tilde{b}_{TM} : M \to BO_n$ to a map $\hat{b}_{TM} : M \to BSO_n$ is the same as defining an SO_n structure on the stablized tangent bundle $\tilde{T}M = \underline{\mathbb{R}}^{n-r} \oplus TM$, which is the same as defining an orientation of $\tilde{T}M$, since $\underline{\mathbb{R}}^{n-r}$ has a canonical orientation.

A stable framing is the tangential structure given by the universal bundle $EO \to BO$. The bullback of this to the BO_n's givens the universal frame bundle $EO_n \to BO_n$. A lift of the classifying map of $\tilde{T}M$ from BO_n to EO_n is the same as defining a section of the frame bundle of $\tilde{T}M$, which is the same as defining a trivialization of $\tilde{T}M$.

The above examples show that a tangential structure is just fancy way of encoding that certain geometric requirements be placed on the tangent bundle. One can similarly define spin structures, complex structures, etc.

Definition 3.5. We can now define the bordism category $\mathbf{Bord}_n^\Theta$ of smooth manifolds with a given tangential structure $\pi : \Theta \to BO$. The objects will consist of compact smooth closed n-manfiolds with a Θ-structure. The morphisms will be bordisms with Θ-structures that restrict to the boundaries appropriatly. We require the Θ-structures to respect the product structure on the ends given by the θ_i maps, $i = 1, 2$.

If M has a Θ structure then on can extend it levelwise to a Θ structure on $M \times [0, 1]$. This gives the identity morphism for M. Similarly, for composition one can glue the appropriate lifts associated to composable bordisms to get a lift for the composition. This requires that the Θ structure of the bordisms be product-like on the ends.

Example 3.6. Now the we can define the oriented bordism category Bord_n^{SO}, and the framed bordism category $\mathrm{Bord}_n^{\mathrm{fr}}$ simply as the cobordism category with the appropriate tangential structure.

Similarly we can define bordism groups Ω_n^Θ with tangential structure just as we did without the tangetial structure. We note that the inverses still exist since we can extend a tangential structure from M to $M \times [0, 1]$. In this way we get the oriented cobordism groups Ω_n^{SO}, and the framed cobordism groups Ω_n^{fr} for every $n \geq 0$.

Example 3.7. Let us calculate some low dimensional examples. When $n = 0$ things are relatively easy. Every compact 0-dimensional manifold is a finite union of points. A compact 1-dimensional manifolds is a finite union of disjoing intervals along with a finite number of disjoint circles. Thus in the cobordism group $\mathfrak{N}_0$ an object is a diffeomorphism class of a finite collection $x_1, \cdots x_n$ of points, or the empty set $\emptyset$. Thus, since $\mathfrak{N}_0$ is a group, there is a map from $\mathbb{Z}$ onto $\mathfrak{N}_0$, that takes 1 to the class $[pt]$ containing one point. Since the interval can be thought of as a map from $x \sqcup x \to \emptyset$, we have that under this surjective map 2 goes to 0. Thus, since a single point is not the boundary of any 1-manifold, we have that $\mathfrak{N}_0 \cong \mathbb{Z}/2$, generated by the class of a single point.

Since an orientation and a framing of a point are the same thing, we have that $\Omega_0^{SO} = \Omega_0^{\mathbf{fr}}$. An oriented compact 0-manfiold is a finite collection of points labeled with either $+$ or $-$. We note now that the interval is a morphism between $[pt, +]$ and $[pt, -]$. Thus the surjective map given above that takes 1 to $[pt, 1]$ is injective, and we have that $\Omega_0^{SO} = \Omega_0^{\mathbf{fr}} \cong \mathbb{Z}$.

4. Stong's Cobordism Theory

Definition 4.1. A cobordism category is a triple $(\mathcal{C}, \partial, i)$ in which:

(1) $\mathcal{C}$ is a category having finite sums and an initial object;
(2) $\partial : \mathcal{C} \to \mathcal{C}$ is an additive functor such that for each object X of $\mathcal{C}$, $\partial\partial\,(X)$ is the initial object;
(3) $i : \delta \to I$ is a natural transformation of additive functors from ∂ to the identity functor I; and
(4) There is a small category $\mathcal{C}_0$ of $\mathcal{C}$ such that each object of $\mathcal{C}$ is isomorphic to an object of $\mathcal{C}_0$.

Example 4.2. We define a triple $(\mathcal{D}, \partial, i)$ which satisfies the above conditions, and is therefore a cobordism category. The category $\mathcal{D}$ is the category whose objects are smooth compact manifolds, and whose maps are smooth boundary preserving maps. The sum in this category is given by disjoint union, and the initial object is the empty manifold. For each object of $\mathcal{D}$ its boundary is again an object of $\mathcal{D}$, and for any map in $\mathcal{D}$ the restriction to the boundary is again a map in $\mathcal{D}$. This restriction is compatible with composition, the identity maps, and preserves direct sums. Thus we define the functor $\partial : \mathcal{D} \to \mathcal{D}$ to be the functor that takes an object X to its boundary ∂X, and takes a morphism to the restriction of that morphism to the boundary. The boundary ∂X of X is a subset whose inclusion $i_X : \partial X \to X$ is a smooth map. Thus define the natural transformation between the functor ∂ and the identity functor to assign to the object X the inclusion morphism $i_X : \partial X \to X$. The Whitney embedding theorem says that every manifold is diffeomorphic to a submanifold of countable dimensional Euclidean space. Thus the subcategory $\mathcal{D}_0$ of $\mathcal{D}$ consisting of all submanifold of $\mathbb{R}^\infty$ is a small category and every object in $\mathcal{D}$ is isomorphic to an object in $\mathcal{D}_0$.

Example 4.3. Let R be a commutative ring with unit $1 \in R$. let $CH_R^{fg,p}$ be the category of chain complexes of finitely generated projective modules over R with chain complex homomorphisms. The initial object in this category is the chain complex $0_\bullet$ with zero in every degree. The direct sum of chain complexes gives the sum in this category. The functor ∂ is defined as follows $\partial\,(C_\bullet, d) = (Z\,(C_\bullet)\,, 0)$ assigns to the chain complex $C_\bullet$, the kernel of d, $Z\,(C_\bullet)$, with zero differential. The natural transformation $i_C : Z\,(C_\bullet) \to C_\bullet$ is the inclusion of the kernel. One can realize any object in this category as the direct sumamnd of an ojbect all of whose R-modules are free, and the category of chain complexes of free R-modules is a small category.

Definition 4.4. If $(\mathcal{C}, \partial, i)$ is a cobordism category, one says that the objects X and Y of $\mathcal{C}$ are cobordant if there exists objects U and V of $\mathcal{C}$ such that the sum of X and ∂U is isomorphic to the sum of Y and ∂V. This will be writen $X \equiv Y$, and denoted $X \sqcup \partial U \cong Y \sqcup \partial V$.

5. Topological Field Theories

This section follows Atiya's classical paper.

Definition 5.1. A topological quantum field theory in dimesnion d over a ground field R consists of the following data:

(1) A finitely generated R-module $Z(\Sigma)$ associated to each oriented closed smooth d-dimensional manifold Σ, such that $Z(\emptyset) = R$.
(2) An element $z(M) \in Z(\partial M)$ associated to each oriented smooth $(d+1)$-dimensional manifold M.
(3) These data are subject to the following axioms:
  - a1 (a) Z is functorial with respect to orientation preserving diffeomorphisms of Σ and M;
  - a2 (b) Z is involutory, i.e. there is an isomorphism $Z(\Sigma^*) \cong Z(\Sigma)^*$ where Σ^* is Σ with the opposite orientation and $Z(\Sigma)^*$ denotes the dual module;
  - a3 (c) Z is multiplicative.
  - a4 (d) For each oriented closed smooth d-dimensional manifold Σ, when $\Sigma \times [0,1]$ has the orientation induced from that of Σ, we have $z(\Sigma \times [0,1]) \in Z(\Sigma) \otimes Z(\Sigma)^*$ maps to the identity map under the canonical isomorphism $Z(\Sigma)^* \otimes Z(\Sigma) \cong \mathrm{Hom}(Z(\Sigma), Z(\Sigma))$.

We elaborate on these axioms below.

Remark 5.2 (Axiom a1 3a)**.** This axion says that any orientation preserving diffeomorphism $f : \Sigma \to \Sigma'$ induces an isomorphism $Z(f) : Z(\Sigma) \to Z(\Sigma')$ such that if $g : \Sigma' \to \Sigma''$ is another orientation preserving diffeomorphism, then $Z(g \circ f) = Z(g) \circ Z(f)$. Furthermore if $f = F|_{\partial M}$ is the restriction of an orientation preserving diffeomorphism $F : M \to M'$ with $\partial M = \Sigma$,and $\partial M' = \Sigma'$, then $Z(f)(z(M)) = z(M') \in Z(\Sigma')$.

Remark 5.3 (Axiom a2 3b)**.** We consider only case when R is a field. In this case the vector space $Z(\Sigma^*)$ is the dual vector space $Z(\Sigma)^*$. Because $Z(\Sigma)$ is finitely generated we have $Z(\Sigma) \cong Z(\Sigma)^{**}$, as it should be since $\Sigma^{**} = \Sigma$.

Remark 5.4 (Axiom a3 3c)**.** This is the most important and the most complicated axiom, so we will discuss it in detail. Firstly, this axiom says that for the disjoint union of oriented smooth manfiolds $\Sigma_1 \sqcup \Sigma_2$ we have $Z(\Sigma_1 \sqcup \Sigma_2) = Z(\Sigma_1) \otimes Z(\Sigma_2)$. Consider the case when $\partial M_1 = \Sigma_1 \sqcup \Sigma_2$, $\partial M_2 = \Sigma_2^* \sqcup \Sigma_3$, and M is obtained from gluing M_1 to M_2 along the common boundary Σ_2 via a orientation reversing diffeomorphism on a collar neighborhood, then we require that $z(M) = \langle z(M_1), z(M_2) \rangle$, where this denotes the natural dual pairing on the middle two components

$$Z(\Sigma_1) \otimes Z(\Sigma_2) \otimes Z(\Sigma_2)^* \otimes Z(\Sigma_3) \to Z(\Sigma_1) \otimes Z(\Sigma_3).$$

When $\Sigma_2 = \emptyset$, so that $M = M_1 \sqcup M_2$ then $z(M) = z(M_1) \otimes z(M_2)$.

Remark 5.5. Recal that for any vector spaces A and B over R we have that there canonical isomorphisms

$$\alpha : A^* \otimes B \to \mathrm{Hom}(A, B) : \alpha(f \otimes b)(a) = f(a) \cdot b,$$

and

$$p : A^* \otimes B \to \mathrm{Hom}(R, A^* \otimes B) : p(f \otimes b)(r) = r \cdot (f \otimes b).$$

The map p has an inverse $u(h) = h(1)$.

Since B is finite-dimensional we have that the evaluation map $b \mapsto ev_b : B^* \to R$ from B to B^{**} is an isomorphism. By composing with this map we get an isomorphism

$$e : \text{Hom}(A, B) \to \text{Hom}(A, B^{**}) : e(g)(a) = ev_{g(a)}.$$

We also have an isomorphism

$$\beta : \text{Hom}(A \otimes B^*, R) \to \text{Hom}(A, B^{**}) : (\beta(H)(a))(g) = H(a \otimes g)$$

Under these isomorphisms we see that given any compact smooth manifold M whose boundary is $\Sigma_1^* \sqcup \Sigma_2$, the element $z(M) \in Z(\Sigma_1)^* \otimes Z(\Sigma_2)$ can be identified with a map $\alpha z(M) \in \text{Hom}(Z(\Sigma_1), Z(\Sigma_2))$, a map $pz(M) \in \text{Hom}(R, A^* \otimes B)$, or a map $\beta e \alpha z(M) \in \text{Hom}(A \otimes B^*, R)$. We will abuse notation and use the same symbol for all three elemets. The rule that $z(M) = \langle z(M_1), z(M_2) \rangle$ when $M = M_1 \cup_{\Sigma_2} M_2$ translates under these isomorphisms to composition of function. That is to say under this isomorphism we have $z(M) = z(M_2) \circ z(M_1)$ when thought of as functions. This composition is associative since the above pairings are associative. Therefore, one can view a bordism from Σ_1 to Σ_2 as inducing a morphism from $Z(\Sigma_1) \to Z(\Sigma_2)$, a morphism $R \to Z(\Sigma_1)^* \otimes Z(\Sigma_2)$, and a map $Z(\Sigma_1) \otimes Z(\Sigma_2)^* \to R$.

Notice that if $M = \Sigma \times [0,1]$ the $\partial M = \Sigma^* \sqcup \Sigma$. So this induces the morphisms

$$z_I(M) : Z(\Sigma) \to Z(\Sigma);$$

$$z_-(M) : R \to Z(\Sigma)^* \otimes Z(\Sigma);$$

and

$$z_+(M) : Z(\Sigma) \otimes Z(\Sigma)^* \to R.$$

Here $z_I(M) = id_{Z(M)}$, and if we glue M to itself along its boundary we get a closed manifold $S^1 \times M$ whose invariant satisfies

$$z\left(S^1 \times \Sigma\right) = z_+(M) \circ z_-(M) \in \text{End}(R).$$

This map is determined by its value on 1, and an easy calculation shows that

$$z\left(S^1 \times \Sigma\right) = \dim_R Z(\Sigma).$$

Example 5.6. Let M be a smooth compact n-dimensional manifold. Let F be a finite group, and $Bun_F(M)$ the collection of principle F-bundles on M. Let $Z_F(M)$ be the collection of functions $Bun_F(M) \to \mathbb{Z}$. If W is an $n+1$-dimensional bordism from $\partial_- W$ to $\partial_+ W$. Define the morphism $z_F(W) : Z_F(\partial_- W) \to Z_F(\partial_+ W)$ by assigning to a function $f : Bun_F(\partial_- W) \to \mathbb{Z}$, the function $z_F(W)(f) : Bun_F(\partial_F W) \to \mathbb{Z}$ which when applied to the principle F bundle ξ over $\partial_+ W$ gives the output

$$\frac{1}{|F|} \sum_{i_+^* E \cong \xi} f\left(i_-^* E\right),$$

where $i_\pm^* : Bun_F(W) \to Bun_F(\partial_\pm W)$ is the restriction of bundles induced by the appropriate inclusion, and the sum varies over all $E \in Bun_F(W)$ such that $i_+^* E \cong \xi$. Notice that the above sum is an integer since for every E such that $i_+^* E \cong \xi$ every element of F defines an automorphism of ξ, thereby giving a different isomorphis $i_+^* E \cong \xi$. Thus fore each $E \in Bun_F(W)$ there are are either 0, or $|F|$ different isomorhpisms $i_+^* E \cong \xi$.

Thus for a closed $n+1$-dimensional manifold W the invariant $z_F(W)$ is

$$z_F(W) = \frac{|Bun_F(W)|}{|F|}$$

In this way we define an n-dimensional TFT associated to the finite group F. The only thing to check is that the composition law holds true, i.e. if $W = W_1 \cup_M W_2$ is obtained from gluing two bordisms W_1 and W_2 along the outgoing end $\partial_+ W_1 \cong M$ and the incoming end $\partial_- W_2 \cong M$ of W_2. then we have the equality

$$z_F(W) = z_F(W_2) \circ z_F(W_1).$$

So for a function $f : Bun_F(\partial_- W_1) \to \mathbb{Z}$, we can define a function $Bun_F(\partial_+ W_2) \to \mathbb{Z}$ in two ways. First we have

$$z_F(W)(f)(\xi) = \frac{1}{|F|} \sum_{i_+^* E \cong \xi} f\left(i_-^* E\right).$$

and secondly we have

$$z_F(W_2) \circ z_F(W_1)(f)(\xi) = \frac{1}{|F|} \sum_{i_{2,+}^* C \cong \xi} z_F(W_1)(f)\left(i_{2,-}^* C\right)$$

this then becomes

$$\frac{1}{|F|} \sum_{i_{2,+}^* C \cong \xi} \frac{1}{|F|} \sum_{i_{1,+}^* D \cong i_{2,-}^* C} f\left(i_{1,-}^* D\right).$$

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